FORMALIZED MATHEMATICS
Volume 9, Number **2**, 2001
University of Białystok

Basic Properties of Fuzzy Set Operation and Membership Function

Takashi Mitsuishi
Shinshu University
Nagano

Katsumi Wasaki
Shinshu University
Nagano

Yasunari Shidama
Shinshu University
Nagano

Summary. This article introduces the fuzzy theory. The definition of the difference set, algebraic product and algebraic sum of fuzzy set is shown. In addition, basic properties of those operations are described. Basic properties of fuzzy set are a little different from those of crisp set.

MML Identifier: `FUZZY_2`.

The articles [3], [1], [2], [4], and [5] provide the terminology and notation for this paper.

1. Basic Properties of Membership Function and Difference Set

For simplicity, we follow the rules: C denotes a non empty set, c denotes an element of C, f, h, g, h_1 denote membership functions of C, A denotes a FuzzySet of C, f, B denotes a FuzzySet of C, g, D denotes a FuzzySet of C, h, D_1 denotes a FuzzySet of C, h_1, X denotes a Universal FuzzySet of C, and E denotes an Empty FuzzySet of C.

We now state four propositions:

(1) For every element x of C and for every membership function h of C holds $0 \leqslant h(x)$ and $h(x) \leqslant 1$.

(2) For every element x of C holds $(\mathrm{EMF}\,C)(x) = 0$ and $(\mathrm{UMF}\,C)(x) = 1$.

(3) For every c such that $f(c) \leqslant h(c)$ holds $(\max(f, \min(g, h)))(c) = (\min(\max(f, g), h))(c)$.

(4) If $A \subseteq D$, then $A \cup B \cap D = (A \cup B) \cap D$.

ISSN 1426–2630

Let C be a non empty set, let f, g be membership functions of C, let A be a FuzzySet of C, f, and let B be a FuzzySet of C, g. The functor $A \setminus B$ yielding a FuzzySet of C, $\min(f, 1\text{-minus}\, g)$ is defined as follows:

(Def. 1) $A \setminus B = [\!: C, (\min(f, 1\text{-minus}\, g))^{\circ} C :\!]$.

Next we state a number of propositions:

(5) $A \setminus B = A \cap B^{\rm c}$.

(6) $1\text{-minus} \min(f, 1\text{-minus}\, g) = \max(1\text{-minus}\, f, g)$.

(7) $(A \setminus B)^{\rm c} = A^{\rm c} \cup B$.

(8) For every c such that $f(c) \leqslant g(c)$ holds $(\min(f, 1\text{-minus}\, h))(c) \leqslant (\min(g, 1\text{-minus}\, h))(c)$.

(9) If $A \subseteq B$, then $A \setminus D \subseteq B \setminus D$.

(10) For every c such that $f(c) \leqslant g(c)$ holds $(\min(h, 1\text{-minus}\, g))(c) \leqslant (\min(h, 1\text{-minus}\, f))(c)$.

(11) If $A \subseteq B$, then $D \setminus B \subseteq D \setminus A$.

(12) For every c such that $f(c) \leqslant g(c)$ and $h(c) \leqslant h_1(c)$ holds $(\min(f, 1\text{-minus}\, h_1))(c) \leqslant (\min(g, 1\text{-minus}\, h))(c)$.

(13) If $A \subseteq B$ and $D \subseteq D_1$, then $A \setminus D_1 \subseteq B \setminus D$.

(14) For every c holds $(\min(f, 1\text{-minus}\, g))(c) \leqslant f(c)$.

(15) $A \setminus B \subseteq A$.

(16) For every c holds $(\min(f, 1\text{-minus}\, g))(c) \leqslant (\max(\min(f, 1\text{-minus}\, g), \min(1\text{-minus}\, f, g)))(c)$.

(17) $A \setminus B \subseteq A \dot{-} B$.

(18) $A \setminus E = A$.

(19) $E \setminus A = E$.

(20) For every c holds $(\min(f, 1\text{-minus}\, g))(c) \leqslant (\min(f, 1\text{-minus} \min(f, g)))(c)$.

(21) $A \setminus B \subseteq A \setminus A \cap B$.

(22) For every c holds $(\max(\min(f, g), \min(f, 1\text{-minus}\, g)))(c) \leqslant f(c)$.

(23) For every c holds $(\max(f, \min(g, 1\text{-minus}\, f)))(c) \leqslant (\max(f, g))(c)$.

(24) $A \cup (B \setminus A) \subseteq A \cup B$.

(25) $A \cap B \cup (A \setminus B) \subseteq A$.

(26) $\min(f, 1\text{-minus} \min(g, 1\text{-minus}\, h)) = \max(\min(f, 1\text{-minus}\, g), \min(f, h))$.

(27) $A \setminus (B \setminus D) = (A \setminus B) \cup A \cap D$.

(28) For every c holds $(\min(f, g))(c) \leqslant (\min(f, 1\text{-minus} \min(f, 1\text{-minus}\, g)))(c)$.

(29) $A \cap B \subseteq A \setminus (A \setminus B)$.

(30) For every c holds $(\min(f, 1\text{-minus}\, g))(c) \leqslant (\min(\max(f, g), 1\text{-minus}\, g))(c)$.

(31) $A \setminus B \subseteq (A \cup B) \setminus B$.

(32) $\min(f, 1\text{-minus} \max(g, h)) = \min(\min(f, 1\text{-minus}\, g), \min(f, 1\text{-minus}\, h))$.

(33) $A \setminus (B \cup D) = (A \setminus B) \cap (A \setminus D)$.

(34) $\min(f, 1\text{-minus}\min(g, h)) = \max(\min(f, 1\text{-minus}\, g), \min(f, 1\text{-minus}\, h))$.

(35) $A \setminus B \cap D = (A \setminus B) \cup (A \setminus D)$.

(36) $\min(\min(f, 1\text{-minus}\, g), 1\text{-minus}\, h) = \min(f, 1\text{-minus}\max(g, h))$.

(37) $A \setminus B \setminus D = A \setminus (B \cup D)$.

(38) For every c holds $(\min(\max(f, g), 1\text{-minus}\min(f, g)))(c) \geqslant (\max(\min(f, 1\text{-minus}\, g), \min(g, 1\text{-minus}\, f)))(c)$.

(39) $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus A \cap B$.

(40) $\min(\max(f, g), 1\text{-minus}\, h) = \max(\min(f, 1\text{-minus}\, h), \min(g, 1\text{-minus}\, h))$.

(41) $(A \cup B) \setminus D = (A \setminus D) \cup (B \setminus D)$.

(42) For every c such that $(\min(f, 1\text{-minus}\, g))(c) \leqslant h(c)$ and $(\min(g, 1\text{-minus}\, f))(c) \leqslant h(c)$ holds $(\max(\min(f, 1\text{-minus}\, g), \min(1\text{-minus}\, f, g)))(c) \leqslant h(c)$.

(43) If $A \setminus B \subseteq D$ and $B \setminus A \subseteq D$, then $A \dot{-} B \subseteq D$.

(44) $A \cap (B \setminus D) = A \cap B \setminus D$.

(45) For every c holds $(\min(f, \min(g, 1\text{-minus}\, h)))(c) \leqslant (\min(\min(f, g), 1\text{-minus}\min(f, h)))(c)$.

(46) $A \cap (B \setminus D) \subseteq A \cap B \setminus A \cap D$.

(47) For every c holds $(\min(\max(f, g), 1\text{-minus}\min(f, g)))(c) \geqslant (\max(\min(f, 1\text{-minus}\, g), \min(1\text{-minus}\, f, g)))(c)$.

(48) $A \dot{-} B \subseteq (A \cup B) \setminus A \cap B$.

(49) For every c holds $(\max(\min(f, g), 1\text{-minus}\max(f, g)))(c) \leqslant (1\text{-minus}\max(\min(f, 1\text{-minus}\, g), \min(1\text{-minus}\, f, g)))(c)$.

(50) $A \cap B \cup (A \cup B)^{\mathrm{c}} \subseteq (A \dot{-} B)^{\mathrm{c}}$.

(51) $\min(\max(\min(f, 1\text{-minus}\, g), \min(1\text{-minus}\, f, g)), 1\text{-minus}\, h) = \max(\min(f, 1\text{-minus}\max(g, h)), \min(g, 1\text{-minus}\max(f, h)))$.

(52) $(A \dot{-} B) \setminus D = (A \setminus (B \cup D)) \cup (B \setminus (A \cup D))$.

(53) For every c holds $(\min(f, 1\text{-minus}\max(\min(g, 1\text{-minus}\, h), \min(1\text{-minus}\, g, h))))(c) \geqslant (\max(\min(f, 1\text{-minus}\max(g, h)), \min(\min(f, g), h)))(c)$.

(54) $(A \setminus (B \cup D)) \cup A \cap B \cap D \subseteq A \setminus (B \dot{-} D)$.

(55) For every c such that $f(c) \leqslant g(c)$ holds $g(c) \geqslant (\max(f, \min(g, 1\text{-minus}\, f)))(c)$.

(56) If $A \subseteq B$, then $A \cup (B \setminus A) \subseteq B$.

(57) For every c holds $(\max(f, g))(c) \geqslant (\max(\max(\min(f, 1\text{-minus}\, g), \min(1\text{-minus}\, f, g)), \min(f, g)))(c)$.

(58) $(A \dot{-} B) \cup A \cap B \subseteq A \cup B$.

(59) If $\min(f, 1\text{-minus}\, g) = \mathrm{EMF}\, C$, then for every c holds $f(c) \leqslant g(c)$.

(60) If $A \setminus B = E$, then $A \subseteq B$.

(61) If $\min(f, g) = \mathrm{EMF}\, C$, then $\min(f, 1\text{-minus}\, g) = f$.

(62) If $A \cap B = E$, then $A \setminus B = A$.

2. Algebraic Product and Algebraic Sum

Let C be a non empty set and let h, g be membership functions of C. The functor $h \cdot g$ yielding a membership function of C is defined as follows:

(Def. 2) For every element c of C holds $(h \cdot g)(c) = h(c) \cdot g(c)$.

Let C be a non empty set and let h, g be membership functions of C. The functor $h \oplus g$ yielding a membership function of C is defined as follows:

(Def. 3) For every element c of C holds $(h \oplus g)(c) = (h(c) + g(c)) - h(c) \cdot g(c)$.

Let C be a non empty set, let h, g be membership functions of C, let A be a FuzzySet of C, h, and let B be a FuzzySet of C, g. The functor $A \cdot B$ yields a FuzzySet of C, $h \cdot g$ and is defined as follows:

(Def. 4) $A \cdot B = [\!: C, (h \cdot g)^{\circ} C :\!]$.

Let C be a non empty set, let h, g be membership functions of C, let A be a FuzzySet of C, h, and let B be a FuzzySet of C, g. The functor $A \oplus B$ yielding a FuzzySet of C, $h \oplus g$ is defined by:

(Def. 5) $A \oplus B = [\!: C, (h \oplus g)^{\circ} C :\!]$.

We now state a number of propositions:

(63) For every c holds $(f \cdot f)(c) \leqslant f(c)$ and $(f \oplus f)(c) \geqslant f(c)$.

(64) $A \cdot A \subseteq A$ and $A \subseteq A \oplus A$.

(65) $f \cdot g = g \cdot f$ and $f \oplus g = g \oplus f$.

(66) $A \cdot B = B \cdot A$ and $A \oplus B = B \oplus A$.

(67) $(f \cdot g) \cdot h = f \cdot (g \cdot h)$.

(68) $(A \cdot B) \cdot D = A \cdot (B \cdot D)$.

(69) $(f \oplus g) \oplus h = f \oplus (g \oplus h)$.

(70) $(A \oplus B) \oplus D = A \oplus (B \oplus D)$.

(71) For every c holds $(f \cdot (f \oplus g))(c) \leqslant f(c)$ and $(f \oplus f \cdot g)(c) \geqslant f(c)$.

(72) $A \cdot (A \oplus B) \subseteq A$ and $A \subseteq A \oplus A \cdot B$.

(73) For every c holds $(f \cdot (g \oplus h))(c) \leqslant (f \cdot g \oplus f \cdot h)(c)$.

(74) $A \cdot (B \oplus D) \subseteq A \cdot B \oplus A \cdot D$.

(75) For every c holds $((f \oplus g) \cdot (f \oplus h))(c) \leqslant (f \oplus g \cdot h)(c)$.

(76) $(A \oplus B) \cdot (A \oplus D) \subseteq A \oplus B \cdot D$.

(77) $\text{1-minus}\, f \cdot g = \text{1-minus}\, f \oplus \text{1-minus}\, g$.

(78) $(A \cdot B)^{\mathrm{c}} = A^{\mathrm{c}} \oplus B^{\mathrm{c}}$.

(79) $\text{1-minus}\, f \oplus g = \text{1-minus}\, f \cdot \text{1-minus}\, g$.

(80) $(A \oplus B)^{\mathrm{c}} = A^{\mathrm{c}} \cdot B^{\mathrm{c}}$.

(81) $f \oplus g =$ 1-minus 1-minus $f \cdot$ 1-minus g.

(82) $A \oplus B = (A^{\mathrm{c}} \cdot B^{\mathrm{c}})^{\mathrm{c}}$.

(83) $f \cdot \mathrm{EMF}\, C = \mathrm{EMF}\, C$ and $f \cdot \mathrm{UMF}\, C = f$.

(84) $A \cdot E = E$ and $A \cdot X = A$.

(85) $f \oplus \mathrm{EMF}\, C = f$ and $f \oplus \mathrm{UMF}\, C = \mathrm{UMF}\, C$.

(86) $A \oplus E = A$ and $A \oplus X = X$.

(87) For every c holds $(\mathrm{EMF}\, C)(c) \leqslant (f \cdot \text{1-minus}\, f)(c)$.

(88) For every c holds $(\mathrm{UMF}\, C)(c) \geqslant (f \oplus \text{1-minus}\, f)(c)$.

(89) $E \subseteq A \cdot A^{\mathrm{c}}$ and $A \oplus A^{\mathrm{c}} \subseteq X$.

(90) For every c holds $(f \cdot g)(c) \leqslant (\min(f, g))(c)$.

(91) $A \cdot B \subseteq A \cap B$.

(92) For every c holds $(\max(f, g))(c) \leqslant (f \oplus g)(c)$.

(93) $A \cup B \subseteq A \oplus B$.

(94) For all real numbers a, b, c such that $0 \leqslant c$ holds $c \cdot \max(a, b) = \max(c \cdot a, c \cdot b)$ and $c \cdot \min(a, b) = \min(c \cdot a, c \cdot b)$.

(95) For all real numbers a, b, c holds $c + \max(a, b) = \max(c + a, c + b)$ and $c + \min(a, b) = \min(c + a, c + b)$.

(96) $f \cdot \max(g, h) = \max(f \cdot g, f \cdot h)$.

(97) $f \cdot \min(g, h) = \min(f \cdot g, f \cdot h)$.

(98) $A \cdot (B \cap D) = (A \cdot B) \cap (A \cdot D)$ and $A \cdot (B \cup D) = A \cdot B \cup A \cdot D$.

(99) $f \oplus \max(g, h) = \max(f \oplus g, f \oplus h)$.

(100) $f \oplus \min(g, h) = \min(f \oplus g, f \oplus h)$.

(101) $A \oplus (B \cup D) = (A \oplus B) \cup (A \oplus D)$ and $A \oplus B \cap D = (A \oplus B) \cap (A \oplus D)$.

(102) For every c holds $(\max(f, g) \cdot \max(f, h))(c) \leqslant (\max(f, g \cdot h))(c)$.

(103) For every c holds $(\min(f, g) \cdot \min(f, h))(c) \leqslant (\min(f, g \cdot h))(c)$.

(104) $(A \cup B) \cdot (A \cup D) \subseteq A \cup B \cdot D$ and $(A \cap B) \cdot (A \cap D) \subseteq A \cap (B \cdot D)$.

(105) For every element c of C and for all membership functions f, g of C holds $(f \oplus g)(c) = 1 - (1 - f(c)) \cdot (1 - g(c))$.

(106) For every c holds $(\max(f, g \oplus h))(c) \leqslant (\max(f, g) \oplus \max(f, h))(c)$.

(107) For every c holds $(\min(f, g \oplus h))(c) \leqslant (\min(f, g) \oplus \min(f, h))(c)$.

(108) $A \cup (B \oplus D) \subseteq (A \cup B) \oplus (A \cup D)$ and $A \cap (B \oplus D) \subseteq A \cap B \oplus A \cap D$.

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Received May 22, 2000