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Injective Spaces. Part II[1]

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The notation and terminology used in this paper are introduced in the following articles: [11], [8], [6], [1], [19], [23], [10], [17], [18], [24], [9], [26], [22], [14], [12], [3], [7], [15], [4], [16], [2], [13], [25], [21], [20], and [5].

1. Injective Spaces

The following propositions are true:

(1) For every point p of the Sierpiński space such that $p = 0$ holds $\{p\}$ is closed.

(2) For every point p of the Sierpiński space such that $p = 1$ holds $\{p\}$ is non closed.

Let us note that the Sierpiński space is non T_1.

One can check that every top-lattice which is complete and Scott is also discernible.

Let us observe that there exists a T_0-space which is injective and strict.

Let us observe that there exists a top-lattice which is complete, Scott, and strict.

Next we state several propositions:

(3) Let I be a non empty set and T be a Scott topological augmentation of $\prod(I \longmapsto 2^1_\subseteq)$. Then the carrier of T = the carrier of $\prod(I \longmapsto$ the Sierpiński space).

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(4) Let L_1, L_2 be complete lattices, T_1 be a Scott topological augmentation of L_1, T_2 be a Scott topological augmentation of L_2, h be a map from L_1 into L_2, and H be a map from T_1 into T_2. If $h = H$ and h is isomorphic, then H is a homeomorphism.

(5) Let L_1, L_2 be complete lattices, T_1 be a Scott topological augmentation of L_1, and T_2 be a Scott topological augmentation of L_2. If L_1 and L_2 are isomorphic, then T_1 and T_2 are homeomorphic.

(6) Let S, T be non empty topological spaces. If S is injective and S and T are homeomorphic, then T is injective.

(7) Let L_1, L_2 be complete lattices, T_1 be a Scott topological augmentation of L_1, and T_2 be a Scott topological augmentation of L_2. If L_1 and L_2 are isomorphic and T_1 is injective, then T_2 is injective.

Let X, Y be non empty topological spaces. Let us observe that X is a topological retract of Y if and only if:

(Def. 1) There exists a continuous map c from X into Y and there exists a continuous map r from Y into X such that $r \cdot c = \mathrm{id}_X$.

One can prove the following propositions:

(8) Let S, T, X, Y be non empty topological spaces. Suppose that
(i) the topological structure of $S =$ the topological structure of T,
(ii) the topological structure of $X =$ the topological structure of Y, and
(iii) S is a topological retract of X.
Then T is a topological retract of Y.

(9) Let T, S_1, S_2 be non empty topological structures. Suppose S_1 and S_2 are homeomorphic and S_1 is a topological retract of T. Then S_2 is a topological retract of T.

(10) Let S, T be non empty topological spaces. Suppose T is injective and S is a topological retract of T. Then S is injective.

(11) Let J be an injective non empty topological space and Y be a non empty topological space. If J is a subspace of Y, then J is a topological retract of Y.

(12) For every complete continuous lattice L holds every Scott topological augmentation of L is injective.

Let L be a complete continuous lattice. Observe that every topological augmentation of L which is Scott is also injective.

Let T be an injective non empty topological space. Note that the topological structure of T is injective.

2. Specialization Order

Let T be a topological structure. The functor ΩT yielding a strict FR-structure is defined by the conditions (Def. 2).

(Def. 2)(i) The topological structure of ΩT = the topological structure of T, and

(ii) for all elements x, y of ΩT holds $x \leqslant y$ iff there exists a subset Y of T such that $Y = \{y\}$ and $x \in \overline{Y}$.

Let T be an empty topological structure. Observe that ΩT is empty.

Let T be a non empty topological structure. Note that ΩT is non empty.

Let T be a topological space. Note that ΩT is topological space-like.

Let T be a topological structure. One can verify that ΩT is reflexive.

Let T be a topological structure. One can verify that ΩT is transitive.

Let T be a T_0-space. One can verify that ΩT is antisymmetric.

One can prove the following propositions:

(13) Let S, T be topological spaces. Suppose the topological structure of S = the topological structure of T. Then $\Omega S = \Omega T$.

(14) Let M be a non empty set and T be a non empty topological space. Then the relational structure of $\Omega \prod(M \longmapsto T)$ = the relational structure of $\prod(M \longmapsto \Omega T)$.

(15) For every Scott complete top-lattice S holds ΩS = the FR-structure of S.

(16) Let L be a complete lattice and S be a Scott topological augmentation of L. Then the relational structure of ΩS = the relational structure of L.

Let S be a Scott complete top-lattice. Note that ΩS is complete.

We now state four propositions:

(17) Let T be a non empty topological space and S be a non empty subspace of T. Then ΩS is a full relational substructure of ΩT.

(18) Let S, T be topological spaces, h be a map from S into T, and g be a map from ΩS into ΩT. If $h = g$ and h is a homeomorphism, then g is isomorphic.

(19) For all topological spaces S, T such that S and T are homeomorphic holds ΩS and ΩT are isomorphic.

(20) For every injective T_0-space T holds ΩT is a complete continuous lattice.

Let T be an injective T_0-space. One can verify that ΩT is complete and continuous.

We now state the proposition

(21) For all non empty topological spaces X, Y holds every continuous map from ΩX into ΩY is monotone.

Let X, Y be non empty topological spaces. Note that every map from ΩX into ΩY which is continuous is also monotone.

Next we state the proposition

(22) For every non empty topological space T and for every element x of the carrier of ΩT holds $\overline{\{x\}} = \downarrow x$.

Let T be a non empty topological space and let x be an element of the carrier of ΩT. One can verify that $\overline{\{x\}}$ is non empty lower and directed and $\downarrow x$ is closed.

Next we state the proposition

(23) For every topological space X holds every open subset of ΩX is upper.

Let T be a topological space. One can verify that every subset of ΩT which is open is also upper.

Let I be a non empty set, let S, T be non empty relational structures, let N be a net in T, and let i be an element of I. Let us assume that the carrier of $T \subseteq$ the carrier of S^I. The functor commute(N, i, S) yielding a strict net in S is defined by the conditions (Def. 3).

(Def. 3)(i) The relational structure of commute(N, i, S) = the relational structure of N, and

(ii) the mapping of commute(N, i, S) = (commute(the mapping of N))(i).

Next we state two propositions:

(24) Let X, Y be non empty topological spaces, N be a net in $[X \to \Omega Y]$, i be an element of the carrier of N, and x be a point of X. Then (the mapping of commute$(N, x, \Omega Y)$)(i) = (the mapping of N)$(i)(x)$.

(25) Let X, Y be non empty topological spaces, N be an eventually-directed net in $[X \to \Omega Y]$, and x be a point of X. Then commute$(N, x, \Omega Y)$ is eventually-directed.

Let X, Y be non empty topological spaces, let N be an eventually-directed net in $[X \to \Omega Y]$, and let x be a point of X. One can verify that commute$(N, x, \Omega Y)$ is eventually-directed.

Let X, Y be non empty topological spaces. Observe that every net in $[X \to \Omega Y]$ is function yielding.

Next we state the proposition

(26) Let X be a non empty topological space, Y be a T_0-space, and N be a net in $[X \to \Omega Y]$. Suppose that for every point x of X holds sup commute$(N, x, \Omega Y)$ exists. Then sup rng (the mapping of N) exists in $(\Omega Y)^{\text{the carrier of } X}$.

3. Monotone Convergence Topological Spaces

Let T be a non empty topological space. We say that T is monotone convergence if and only if the condition (Def. 4) is satisfied.

(Def. 4) Let D be a non empty directed subset of ΩT. Then sup D exists in ΩT and for every open subset V of T such that $\sup D \in V$ holds D meets V.

One can prove the following proposition

(27) Let S, T be non empty topological spaces. Suppose the topological structure of $S =$ the topological structure of T and S is monotone convergence. Then T is monotone convergence.

Let us observe that every T_0-space which is trivial is also monotone convergence.

Let us observe that there exists a topological space which is strict, trivial, and non empty.

One can prove the following two propositions:

(28) Let S be a monotone convergence T_0-space and T be a T_0-space. If S and T are homeomorphic, then T is monotone convergence.

(29) Every Scott complete top-lattice is monotone convergence.

Let L be a complete lattice. One can check that every Scott topological augmentation of L is monotone convergence.

Let L be a complete lattice and let S be a Scott topological augmentation of L. One can check that the topological structure of S is monotone convergence.

We now state the proposition

(30) For every monotone convergence T_0-space X holds ΩX is up-complete.

Let X be a monotone convergence T_0-space. Observe that ΩX is up-complete.

One can prove the following three propositions:

(31) Let X be a monotone convergence non empty topological space and N be an eventually-directed prenet over ΩX. Then sup N exists.

(32) Let X be a monotone convergence non empty topological space and N be an eventually-directed net in ΩX. Then $\sup N \in \operatorname{Lim} N$.

(33) For every monotone convergence non empty topological space X holds every eventually-directed net in ΩX is convergent.

Let X be a monotone convergence non empty topological space. Observe that every eventually-directed net in ΩX is convergent.

We now state two propositions:

(34) Let X be a non empty topological space. Suppose that for every eventually-directed net N in ΩX holds sup N exists and $\sup N \in \operatorname{Lim} N$. Then X is monotone convergence.

(35) Let X be a monotone convergence non empty topological space and Y be a T_0-space. Then every continuous map from ΩX into ΩY is directed-sups-preserving.

Let X be a monotone convergence non empty topological space and let Y be a T_0-space. One can check that every map from ΩX into ΩY which is continuous is also directed-sups-preserving.

Next we state four propositions:

(36) Let T be a monotone convergence T_0-space and R be a T_0-space. If R is a topological retract of T, then R is monotone convergence.

(37) Let T be an injective T_0-space and S be a Scott topological augmentation of ΩT. Then the topological structure of $S =$ the topological structure of T.

(38) Every injective T_0-space is compact, locally-compact, and sober.

(39) Every injective T_0-space is monotone convergence.

One can verify that every T_0-space which is injective is also monotone convergence.

One can prove the following propositions:

(40) Let X be a non empty topological space, Y be a monotone convergence T_0-space, N be a net in $[X \to \Omega Y]$, and f, g be maps from X into ΩY. Suppose that
 (i) $f = \bigsqcup_{((\Omega Y)^{\text{the carrier of } X})} \text{rng}\,(\text{the mapping of } N)$,
 (ii) $\sup \text{rng}\,(\text{the mapping of } N)$ exists in $(\Omega Y)^{\text{the carrier of } X}$, and
 (iii) $g \in \text{rng}\,(\text{the mapping of } N)$.

 Then $g \leqslant f$.

(41) Let X be a non empty topological space, Y be a monotone convergence T_0-space, N be a net in $[X \to \Omega Y]$, x be a point of X, and f be a map from X into ΩY. Suppose for every point a of X holds $\text{commute}(N, a, \Omega Y)$ is eventually-directed and $f = \bigsqcup_{((\Omega Y)^{\text{the carrier of } X})} \text{rng}\,(\text{the mapping of } N)$. Then $f(x) = \sup \text{commute}(N, x, \Omega Y)$.

(42) Let X be a non empty topological space, Y be a monotone convergence T_0-space, and N be a net in $[X \to \Omega Y]$. Suppose that for every point x of X holds $\text{commute}(N, x, \Omega Y)$ is eventually-directed. Then $\bigsqcup_{((\Omega Y)^{\text{the carrier of } X})} \text{rng}\,(\text{the mapping of } N)$ is a continuous map from X into Y.

(43) Let X be a non empty topological space and Y be a monotone convergence T_0-space. Then $[X \to \Omega Y]$ is a directed-sups-inheriting relational substructure of $(\Omega Y)^{\text{the carrier of } X}$.

References

[1] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(**5**):719–725, 1991.

[2] Grzegorz Bancerek. Bounds in posets and relational substructures. *Formalized Mathematics*, 6(**1**):81–91, 1997.
[3] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(**1**):93–107, 1997.
[4] Grzegorz Bancerek. The "way-below" relation. *Formalized Mathematics*, 6(**1**):169–176, 1997.
[5] Grzegorz Bancerek. Bases and refinements of topologies. *Formalized Mathematics*, 7(**1**):35–43, 1998.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[7] Czesław Byliński. Galois connections. *Formalized Mathematics*, 6(**1**):131–143, 1997.
[8] Agata Darmochwał. Compact spaces. *Formalized Mathematics*, 1(**2**):383–386, 1990.
[9] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[10] Adam Grabowski. On the category of posets. *Formalized Mathematics*, 5(**4**):501–505, 1996.
[11] Adam Grabowski. Properties of the product of compact topological spaces. *Formalized Mathematics*, 8(**1**):55–59, 1999.
[12] Adam Grabowski. Scott-continuous functions. Part II. *Formalized Mathematics*, 9(**1**):5–11, 2001.
[13] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Formalized Mathematics*, 6(**1**):117–121, 1997.
[14] Jarosław Gryko. Injective spaces. *Formalized Mathematics*, 7(**1**):57–62, 1998.
[15] Artur Korniłowicz. Meet–continuous lattices. *Formalized Mathematics*, 6(**1**):159–167, 1997.
[16] Artur Korniłowicz. On the topological properties of meet-continuous lattices. *Formalized Mathematics*, 6(**2**):269–277, 1997.
[17] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Formalized Mathematics*, 5(**2**):167–172, 1996.
[18] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[19] Andrzej Trybulec. Natural transformations. Discrete categories. *Formalized Mathematics*, 2(**4**):467–474, 1991.
[20] Andrzej Trybulec. Baire spaces, Sober spaces. *Formalized Mathematics*, 6(**2**):289–294, 1997.
[21] Andrzej Trybulec. Moore-Smith convergence. *Formalized Mathematics*, 6(**2**):213–225, 1997.
[22] Andrzej Trybulec. Scott topology. *Formalized Mathematics*, 6(**2**):311–319, 1997.
[23] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[24] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[25] Mariusz Żynel and Czesław Byliński. Properties of relational structures, posets, lattices and maps. *Formalized Mathematics*, 6(**1**):123–130, 1997.
[26] Mariusz Żynel and Adam Guzowski. T_0 topological spaces. *Formalized Mathematics*, 5(**1**):75–77, 1996.

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