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Irrationality of e

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Summary. We prove the irrationality of square roots of prime numbers and of the number e. In order to be able to prove the last, a proof is given that `number_e = exp(1)` as defined in the Mizar library, that is that

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = \sum_{k=0}^{\infty}\frac{1}{k!}$$

MML Identifier: `IRRAT_1`.

The articles [2], [3], [4], [18], [14], [1], [6], [13], [15], [8], [7], [20], [12], [5], [10], [11], [9], [16], [21], [17], and [19] provide the notation and terminology for this paper.

1. Square Roots of Primes are Irrational

For simplicity, we follow the rules: k, n, p, K, N are natural numbers, x, y, e_1 are real numbers, s_1, s_2, s_3 are sequences of real numbers, and s_4 is a finite sequence of elements of $\mathbb{R}$.

Let us consider x. We introduce x is irrational as an antonym of x is rational.

Let us consider x, y. We introduce x^y as a synonym of x^y.

One can prove the following two propositions:

(1) If p is prime, then $\sqrt{p}$ is irrational.

(2) There exist x, y such that x is irrational and y is irrational and x^y is rational.

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2. A proof that $e = e$

The scheme *LambdaRealSeq* deals with a unary functor $\mathcal{F}$ yielding a real number, and states that:

> There exists s_1 such that for every n holds $s_1(n) = \mathcal{F}(n)$ and for all s_2, s_3 such that for every n holds $s_2(n) = \mathcal{F}(n)$ and for every n holds $s_3(n) = \mathcal{F}(n)$ holds $s_2 = s_3$

for all values of the parameter.

Let us consider k. The functor $\mathbf{a}_k$ is a sequence of real numbers and is defined by:

(Def. 1) For every n holds $\mathbf{a}_k(n) = \frac{n-k}{n}$.

Let us consider k. The functor $\mathbf{b}_k$ is a sequence of real numbers and is defined by:

(Def. 2) For every n holds $\mathbf{b}_k(n) = \binom{n}{k} \cdot n^{-k}$.

Let us consider n. The functor $\mathbf{c}_n$ is a sequence of real numbers and is defined as follows:

(Def. 3) For every k holds $\mathbf{c}_n(k) = \binom{n}{k} \cdot n^{-k}$.

Next we state the proposition

(3) $\mathbf{c}_n(k) = \mathbf{b}_k(n)$.

The sequence $\mathbf{d}$ of real numbers is defined as follows:

(Def. 4) For every n holds $\mathbf{d}(n) = (1 + \frac{1}{n})^n$.

The sequence $\mathbf{e}$ of real numbers is defined as follows:

(Def. 5) For every k holds $\mathbf{e}(k) = \frac{1}{k!}$.

We now state a number of propositions:

(4) If $n > 0$, then $n^{-(k+1)} = \frac{n^{-k}}{n}$.

(5) For all real numbers x, y, z, v, w such that $y \neq 0$ and $z \neq 0$ and $v \neq 0$ and $w \neq 0$ holds $\frac{x}{y \cdot z \cdot \frac{v}{w}} = \frac{w}{z} \cdot \frac{x}{y \cdot v}$.

(6) $\binom{n}{k+1} = \frac{n-k}{k+1} \cdot \binom{n}{k}$.

(7) If $n > 0$, then $\mathbf{b}_{k+1}(n) = \frac{1}{k+1} \cdot \mathbf{b}_k(n) \cdot \mathbf{a}_k(n)$.

(8) If $n > 0$, then $\mathbf{a}_k(n) = 1 - \frac{k}{n}$.

(9) $\mathbf{a}_k$ is convergent and $\lim(\mathbf{a}_k) = 1$.

(10) For every s_1 such that for every n holds $s_1(n) = x$ holds s_1 is convergent and $\lim s_1 = x$.

(11) For every n such that $n > 0$ holds $\mathbf{b}_0(n) = 1$.

(12) $\frac{1}{k+1} \cdot \frac{1}{k!} = \frac{1}{(k+1)!}$.

(13) $\mathbf{b}_k$ is convergent and $\lim(\mathbf{b}_k) = \frac{1}{k!}$ and $\lim(\mathbf{b}_k) = \mathbf{e}(k)$.

(14) If $k < n$, then $0 < \mathbf{a}_k(n)$ and $\mathbf{a}_k(n) \leqslant 1$.

(15) If $n > 0$, then $0 \leqslant \mathbf{b}_k(n)$ and $\mathbf{b}_k(n) \leqslant \frac{1}{k!}$ and $\mathbf{b}_k(n) \leqslant \mathbf{e}(k)$ and $0 \leqslant \mathbf{c}_n(k)$ and $\mathbf{c}_n(k) \leqslant \frac{1}{k!}$ and $\mathbf{c}_n(k) \leqslant \mathbf{e}(k)$.

(16) For every s_1 such that $s_1 \uparrow 1$ is summable holds s_1 is summable and $\sum s_1 = s_1(0) + \sum(s_1 \uparrow 1)$.

(17) For every s_4 such that $\operatorname{len} s_4 = n$ and $1 \leqslant k$ and $k < n$ holds $(s_4)_{\lfloor 1}(k) = s_4(k+1)$.

(18) For every s_4 such that $\operatorname{len} s_4 > 0$ holds $\sum s_4 = s_4(1) + \sum((s_4)_{\lfloor 1})$.

(19) Let given n and given s_1, s_4. Suppose $\operatorname{len} s_4 = n$ and for every k such that $k < n$ holds $s_1(k) = s_4(k+1)$ and for every k such that $k \geqslant n$ holds $s_1(k) = 0$. Then s_1 is summable and $\sum s_1 = \sum s_4$.

(20) If $x \neq 0$ and $y \neq 0$ and $k \leqslant n$, then $\langle \binom{n}{0} x^0 y^n, \ldots, \binom{n}{n} x^n y^0 \rangle (k+1) = \binom{n}{k} \cdot x^{n-k} \cdot y^k$.

(21) If $n > 0$ and $k \leqslant n$, then $\mathbf{c}_n(k) = \langle \binom{n}{0} 1^0 (\frac{1}{n})^n, \ldots, \binom{n}{n} 1^n (\frac{1}{n})^0 \rangle (k+1)$.

(22) If $n > 0$, then $\mathbf{c}_n$ is summable and $\sum(\mathbf{c}_n) = (1+\frac{1}{n})^n$ and $\sum(\mathbf{c}_n) = \mathbf{d}(n)$.

(23) $\mathbf{d}$ is convergent and $\lim \mathbf{d} = e$.

(24) $\mathbf{e}$ is summable and $\sum \mathbf{e} = \exp 1$.

(25) Let given K and d_1 be a sequence of real numbers. If for every n holds $d_1(n) = (\sum_{\alpha=0}^{\kappa} (\mathbf{c}_n)(\alpha))_{\kappa \in \mathbb{N}}(K)$, then d_1 is convergent and $\lim d_1 = (\sum_{\alpha=0}^{\kappa} \mathbf{e}(\alpha))_{\kappa \in \mathbb{N}}(K)$.

(26) If s_1 is convergent and $\lim s_1 = x$, then for every e_1 such that $e_1 > 0$ there exists N such that for every n such that $n \geqslant N$ holds $s_1(n) > x - e_1$.

(27) Suppose that

(i) for every e_1 such that $e_1 > 0$ there exists N such that for every n such that $n \geqslant N$ holds $s_1(n) > x - e_1$, and

(ii) there exists N such that for every n such that $n \geqslant N$ holds $s_1(n) \leqslant x$.

Then s_1 is convergent and $\lim s_1 = x$.

(28) If s_1 is summable, then for every e_1 such that $e_1 > 0$ there exists K such that $(\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}}(K) > \sum s_1 - e_1$.

(29) If $n \geqslant 1$, then $\mathbf{d}(n) \leqslant \sum \mathbf{e}$.

(30) If s_1 is summable and for every k holds $s_1(k) \geqslant 0$, then $\sum s_1 \geqslant (\sum_{\alpha=0}^{\kappa} (s_1)(\alpha))_{\kappa \in \mathbb{N}}(K)$.

(31) $\mathbf{d}$ is convergent and $\lim \mathbf{d} = \sum \mathbf{e}$.

e can be characterized by the condition:

(Def. 6) $e = \sum \mathbf{e}$.

e can be characterized by the condition:

(Def. 7) $e = \exp 1$.

3. The Number e is Irrational

We now state a number of propositions:

(32) If x is rational, then there exists n such that $n \geqslant 2$ and $n! \cdot x$ is integer.

(33) $n! \cdot \mathbf{e}(k) = \frac{n!}{k!}$.

(34) $\frac{n!}{k!} > 0$.

(35) If s_1 is summable and for every n holds $s_1(n) > 0$, then $\sum s_1 > 0$.

(36) $n! \cdot \sum(\mathbf{e} \uparrow (n+1)) > 0$.

(37) If $k \leqslant n$, then $\frac{n!}{k!}$ is integer.

(38) $n! \cdot (\sum_{\alpha=0}^{\kappa} \mathbf{e}(\alpha))_{\kappa \in \mathbb{N}}(n)$ is integer.

(39) If $x = \frac{1}{n+1}$, then $\frac{n!}{(n+k+1)!} \leqslant x^{k+1}$.

(40) If $n > 0$ and $x = \frac{1}{n+1}$, then $n! \cdot \sum(\mathbf{e} \uparrow (n+1)) \leqslant \frac{x}{1-x}$.

(41) If $n \geqslant 2$ and $x = \frac{1}{n+1}$, then $\frac{x}{1-x} < 1$.

(42) e is irrational.

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