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Scalar Multiple of Riemann Definite Integral

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Summary. The goal of this article is to prove a scalar multiplicity of Riemann definite integral. Therefore, we defined a scalar product to the subset of real space, and we proved some relating lemmas. At last, we proved a scalar multiplicity of Riemann definite integral. As a result, a linearity of Riemann definite integral was proven by unifying the previous article [7].

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The papers [2], [6], [3], [7], [13], [1], [4], [14], [5], [8], [16], [12], [10], [11], [9], and [15] provide the notation and terminology for this paper.

1. Lemmas of Finite Sequence

We adopt the following rules: r, x, y are real numbers, i, j are natural numbers, and p is a finite sequence of elements of $\mathbb{R}$.

The following proposition is true

(1) For every closed-interval subset A of $\mathbb{R}$ and for every x holds $x \in A$ iff $\inf A \leqslant x$ and $x \leqslant \sup A$.

Let I_1 be a finite sequence of elements of $\mathbb{R}$. We say that I_1 is non-decreasing if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let n be a natural number. Suppose $n \in \operatorname{dom} I_1$ and $n+1 \in \operatorname{dom} I_1$. Let r, s be real numbers. If $r = I_1(n)$ and $s = I_1(n+1)$, then $r \leqslant s$.

One can verify that there exists a finite sequence of elements of $\mathbb{R}$ which is non-decreasing.

The following three propositions are true:

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(2) Let p be a non-decreasing finite sequence of elements of $\mathbb{R}$ and given i, j. If $i \in \operatorname{dom} p$ and $j \in \operatorname{dom} p$ and $i \leqslant j$, then $p(i) \leqslant p(j)$.

(3) Let given p. Then there exists a non-decreasing finite sequence q of elements of $\mathbb{R}$ such that p and q are fiberwise equipotent.

(4) Let D be a non empty set, f be a finite sequence of elements of D, and k_1, k_2, k_3 be natural numbers. If $1 \leqslant k_1$ and $k_3 \leqslant \operatorname{len} f$ and $k_1 \leqslant k_2$ and $k_2 < k_3$, then $(\operatorname{mid}(f, k_1, k_2)) \frown \operatorname{mid}(f, k_2 + 1, k_3) = \operatorname{mid}(f, k_1, k_3)$.

2. Scalar Product of Real Subset

Let X be a subset of $\mathbb{R}$ and let r be a real number. The functor $r \cdot X$ yields a subset of $\mathbb{R}$ and is defined as follows:

(Def. 2) $r \cdot X = \{r \cdot x : x \in X\}$.

The following propositions are true:

(5) Let X, Y be non empty sets and f be a partial function from X to $\mathbb{R}$. If f is upper bounded on X and $Y \subseteq X$, then $f \restriction Y$ is upper bounded on Y.

(6) Let X, Y be non empty sets and f be a partial function from X to $\mathbb{R}$. If f is lower bounded on X and $Y \subseteq X$, then $f \restriction Y$ is lower bounded on Y.

(7) For every non empty subset X of $\mathbb{R}$ holds $r \cdot X$ is non empty.

(8) For every subset X of $\mathbb{R}$ holds $r \cdot X = \{r \cdot x : x \in X\}$.

(9) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded and $0 \leqslant r$ holds $r \cdot X$ is upper bounded.

(10) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded and $r \leqslant 0$ holds $r \cdot X$ is lower bounded.

(11) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded and $0 \leqslant r$ holds $r \cdot X$ is lower bounded.

(12) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded and $r \leqslant 0$ holds $r \cdot X$ is upper bounded.

(13) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded and $0 \leqslant r$ holds $\sup(r \cdot X) = r \cdot \sup X$.

(14) For every non empty subset X of $\mathbb{R}$ such that X is upper bounded and $r \leqslant 0$ holds $\inf(r \cdot X) = r \cdot \sup X$.

(15) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded and $0 \leqslant r$ holds $\inf(r \cdot X) = r \cdot \inf X$.

(16) For every non empty subset X of $\mathbb{R}$ such that X is lower bounded and $r \leqslant 0$ holds $\sup(r \cdot X) = r \cdot \inf X$.

3. Scalar Multiple of Integral

The following propositions are true:

(17) For every non empty set X and for every partial function f from X to $\mathbb{R}$ such that f is total holds $\operatorname{rng}(r\, f) = r \cdot \operatorname{rng} f$.

(18) For all non empty sets X, Z and for every partial function f from X to $\mathbb{R}$ holds $\operatorname{rng}(r\,(f{\restriction}Z)) = r \cdot \operatorname{rng}(f{\restriction}Z)$.

(19) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, and D be an element of divs A. If f is total and bounded on A and $r \geqslant 0$, then $(\text{upper\_sum\_set}\, r\, f)(D) \geqslant r \cdot \inf \operatorname{rng} f \cdot \operatorname{vol}(A)$.

(20) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, and D be an element of divs A. If f is total and bounded on A and $r \leqslant 0$, then $(\text{upper\_sum\_set}\, r\, f)(D) \geqslant r \cdot \sup \operatorname{rng} f \cdot \operatorname{vol}(A)$.

(21) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, and D be an element of divs A. If f is total and bounded on A and $r \geqslant 0$, then $(\text{lower\_sum\_set}\, r\, f)(D) \leqslant r \cdot \sup \operatorname{rng} f \cdot \operatorname{vol}(A)$.

(22) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, and D be an element of divs A. If f is total and bounded on A and $r \leqslant 0$, then $(\text{lower\_sum\_set}\, r\, f)(D) \leqslant r \cdot \inf \operatorname{rng} f \cdot \operatorname{vol}(A)$.

(23) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and given i. Suppose $i \in \operatorname{Seg} \operatorname{len} D$ and f is upper bounded on A and total and $r \geqslant 0$. Then $(\text{upper\_volume}(r\, f, D))(i) = r \cdot (\text{upper\_volume}(f, D))(i)$.

(24) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and given i. Suppose $i \in \operatorname{Seg} \operatorname{len} D$ and f is upper bounded on A and total and $r \leqslant 0$. Then $(\text{lower\_volume}(r\, f, D))(i) = r \cdot (\text{upper\_volume}(f, D))(i)$.

(25) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and given i. Suppose $i \in \operatorname{Seg} \operatorname{len} D$ and f is lower bounded on A and total and $r \geqslant 0$. Then $(\text{lower\_volume}(r\, f, D))(i) = r \cdot (\text{lower\_volume}(f, D))(i)$.

(26) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, D be an element of S, and given i. Suppose $i \in \operatorname{Seg} \operatorname{len} D$ and f is lower bounded on A and total and $r \leqslant 0$. Then $(\text{upper\_volume}(r\, f, D))(i) = r \cdot (\text{lower\_volume}(f, D))(i)$.

(27) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is upper bounded on A and total and $r \geqslant 0$, then $\text{upper\_sum}(r\, f, D) = r \cdot \text{upper\_sum}(f, D)$.

(28) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is upper bounded on A and total and $r \leqslant 0$, then $\mathrm{lower\_sum}(r\,f, D) = r \cdot \mathrm{upper\_sum}(f, D)$.

(29) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is lower bounded on A and total and $r \geqslant 0$, then $\mathrm{lower\_sum}(r\,f, D) = r \cdot \mathrm{lower\_sum}(f, D)$.

(30) Let A be a closed-interval subset of $\mathbb{R}$, f be a partial function from A to $\mathbb{R}$, S be a non empty Division of A, and D be an element of S. If f is lower bounded on A and total and $r \leqslant 0$, then $\mathrm{upper\_sum}(r\,f, D) = r \cdot \mathrm{lower\_sum}(f, D)$.

(31) Let A be a closed-interval subset of $\mathbb{R}$ and f be a partial function from A to $\mathbb{R}$. Suppose f is total and bounded on A and f is integrable on A. Then $r\,f$ is integrable on A and $\mathrm{integral}\, r\,f = r \cdot \mathrm{integral}\, f$.

4. Monotoneity of Integral

One can prove the following propositions:

(32) Let A be a closed-interval subset of $\mathbb{R}$ and f be a partial function from A to $\mathbb{R}$. Suppose f is total and bounded on A and f is integrable on A and for every x such that $x \in A$ holds $f(x) \geqslant 0$. Then $\mathrm{integral}\, f \geqslant 0$.

(33) Let A be a closed-interval subset of $\mathbb{R}$ and f, g be partial functions from A to $\mathbb{R}$. Suppose that

- (i) f is total and bounded on A,
- (ii) f is integrable on A,
- (iii) g is total and bounded on A, and
- (iv) g is integrable on A.

Then $f - g$ is integrable on A and $\mathrm{integral}\, f - g = \mathrm{integral}\, f - \mathrm{integral}\, g$.

(34) Let A be a closed-interval subset of $\mathbb{R}$ and f, g be partial functions from A to $\mathbb{R}$. Suppose that

- (i) f is total and bounded on A,
- (ii) f is integrable on A,
- (iii) g is total and bounded on A,
- (iv) g is integrable on A, and
- (v) for every x such that $x \in A$ holds $f(x) \geqslant g(x)$.

Then $\mathrm{integral}\, f \geqslant \mathrm{integral}\, g$.

5. Definition of Division Sequence

Next we state two propositions:

(35) Let A be a closed-interval subset of $\mathbb{R}$ and f be a partial function from A to $\mathbb{R}$. If f is total and bounded on A, then $\operatorname{rng} \mathrm{upper\_sum\_set} f$ is lower bounded.

(36) Let A be a closed-interval subset of $\mathbb{R}$ and f be a partial function from A to $\mathbb{R}$. If f is total and bounded on A, then $\operatorname{rng} \mathrm{lower\_sum\_set} f$ is upper bounded.

Let A be a closed-interval subset of $\mathbb{R}$. A DivSequence of A is a function from $\mathbb{N}$ into $\operatorname{divs} A$.

Let A be a closed-interval subset of $\mathbb{R}$ and let T be a DivSequence of A. The functor δ_T yielding a sequence of real numbers is defined by:

(Def. 3) For every i holds $\delta_T(i) = \delta_{T(i)}$.

Let A be a closed-interval subset of $\mathbb{R}$, let f be a partial function from A to $\mathbb{R}$, and let T be a DivSequence of A. The functor $\mathrm{upper\_sum}(f, T)$ yields a sequence of real numbers and is defined by:

(Def. 4) For every i holds $(\mathrm{upper\_sum}(f, T))(i) = \mathrm{upper\_sum}(f, T(i))$.

The functor $\mathrm{lower\_sum}(f, T)$ yields a sequence of real numbers and is defined as follows:

(Def. 5) For every i holds $(\mathrm{lower\_sum}(f, T))(i) = \mathrm{lower\_sum}(f, T(i))$.

The following propositions are true:

(37) Let A be a closed-interval subset of $\mathbb{R}$ and D_1, D_2 be elements of $\operatorname{divs} A$. If $D_1 \leqslant D_2$, then for every j such that $j \in \operatorname{dom} D_2$ there exists i such that $i \in \operatorname{dom} D_1$ and $\operatorname{divset}(D_2, j) \subseteq \operatorname{divset}(D_1, i)$.

(38) For all finite non empty subsets X, Y of $\mathbb{R}$ such that $X \subseteq Y$ holds $\max X \leqslant \max Y$.

(39) For all finite non empty subsets X, Y of $\mathbb{R}$ such that there exists y such that $y \in Y$ and $\max X \leqslant y$ holds $\max X \leqslant \max Y$.

(40) For all closed-interval subsets A, B of $\mathbb{R}$ such that $A \subseteq B$ holds $\operatorname{vol}(A) \leqslant \operatorname{vol}(B)$.

(41) For every closed-interval subset A of $\mathbb{R}$ and for all elements D_1, D_2 of $\operatorname{divs} A$ such that $D_1 \leqslant D_2$ holds $\delta_{(D_1)} \geqslant \delta_{(D_2)}$.

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