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The Lawson Topology[1]

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Summary. The article includes definitions, lemmas and theorems 1.1–1.7, 1.9, 1.10 presented in Chapter III of [9, pp. 142–146].

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The articles [20], [15], [14], [8], [6], [1], [18], [13], [19], [17], [3], [11], [4], [12], [2], [10], [16], [5], and [7] provide the notation and terminology for this paper.

1. Lower Topology

Let T be a non empty FR-structure. We say that T is lower if and only if:

(Def. 1) $\{-\uparrow x : x$ ranges over elements of $T\}$ is a prebasis of T.

Let us note that every non empty reflexive topological space-like FR-structure which is trivial is also lower.

One can verify that there exists a top-lattice which is lower, trivial, complete, and strict.

We now state the proposition

(1) For every non empty relational structure L_1 holds there exists a strict correct topological augmentation of L_1 which is lower.

We now state the proposition

(2) Let L_2, L_3 be topological space-like lower non empty FR-structures. Suppose the relational structure of L_2 = the relational structure of L_3. Then the topology of L_2 = the topology of L_3.

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Let R be a non empty relational structure. The functor $\omega(R)$ yielding a family of subsets of R is defined by:

(Def. 2) For every lower correct topological augmentation T of R holds $\omega(R) =$ the topology of T.

Next we state a number of propositions:

(3) Let R_1, R_2 be non empty relational structures. Suppose the relational structure of $R_1 =$ the relational structure of R_2. Then $\omega(R_1) = \omega(R_2)$.

(4) For every lower non empty FR-structure T and for every point x of T holds $-\uparrow x$ is open and $\uparrow x$ is closed.

(5) For every transitive lower non empty FR-structure T and for every subset A of T such that A is open holds A is lower.

(6) For every transitive lower non empty FR-structure T and for every subset A of T such that A is closed holds A is upper.

(7) Let T be a non empty topological space-like FR-structure. Then T is lower if and only if $\{-\uparrow F; F$ ranges over subsets of T: F is finite$\}$ is a basis of T.

(8) Let S, T be lower complete top-lattices and f be a map from S into T. Suppose that for every non empty subset X of S holds f preserves inf of X. Then f is continuous.

(9) Let S, T be lower complete top-lattices and f be a map from S into T. If f is infs-preserving, then f is continuous.

(10) Let T be a lower complete top-lattice, B_1 be a prebasis of T, and F be a non empty filtered subset of T. Suppose that for every subset A of T such that $A \in B_1$ and $\inf F \in A$ holds F meets A. Then $\inf F \in \overline{F}$.

(11) Let S, T be lower complete top-lattices and f be a map from S into T. If f is continuous, then f is filtered-infs-preserving.

(12) Let S, T be lower complete top-lattices and f be a map from S into T. Suppose f is continuous and for every finite subset X of S holds f preserves inf of X. Then f is infs-preserving.

(13) Let T be a lower topological space-like reflexive transitive non empty FR-structure and x be a point of T. Then $\overline{\{x\}} = \uparrow x$.

A top-poset is a topological space-like reflexive transitive antisymmetric FR-structure.

One can check that every non empty top-poset which is lower is also T_0.

Let R be a lower-bounded non empty relational structure. One can verify that every topological augmentation of R is lower-bounded.

We now state four propositions:

(14) Let S, T be non empty relational structures, s be an element of S, and t be an element of T. Then $-\uparrow\langle s, t\rangle = [\!:-\uparrow s,$ the carrier of $T:\!] \cup [\!:$ the carrier of $S, -\uparrow t:\!]$.

(15) Let S, T be lower-bounded non empty posets, S' be a lower correct topological augmentation of S, and T' be a lower correct topological augmentation of T. Then $\omega([\!: S, T :\!]) =$ the topology of $[\!: S'$, (T' **qua** non empty topological space) $:\!]$.

(16) Let S, T be lower lower-bounded non empty top-posets. Then $\omega([\!: S$, (T **qua** poset) $:\!]) =$ the topology of $[\!: S$, (T **qua** non empty topological space) $:\!]$.

(17) Let T, T_2 be lower complete top-lattices. Suppose T_2 is a topological augmentation of $[\!: T$, (T **qua** lattice) $:\!]$. Let f be a map from T_2 into T. If $f = \sqcap_T$, then f is continuous.

2. Refinements Revisited

The scheme *TopInd* deals with a top-lattice $\mathcal{A}$ and and states that:

For every subset A of $\mathcal{A}$ such that A is open holds $\mathcal{P}[A]$

provided the following conditions are met:

- There exists a prebasis K of $\mathcal{A}$ such that for every subset A of $\mathcal{A}$ such that $A \in K$ holds $\mathcal{P}[A]$,
- For every family F of subsets of $\mathcal{A}$ such that for every subset A of $\mathcal{A}$ such that $A \in F$ holds $\mathcal{P}[A]$ holds $\mathcal{P}[\bigcup F]$,
- For all subsets A_1, A_2 of $\mathcal{A}$ such that $\mathcal{P}[A_1]$ and $\mathcal{P}[A_2]$ holds $\mathcal{P}[A_1 \cap A_2]$, and
- $\mathcal{P}[\Omega_{\mathcal{A}}]$.

One can prove the following proposition

(18) Let L_2, L_3 be up-complete antisymmetric non empty reflexive relational structures. Suppose that
  (i) the relational structure of $L_2 =$ the relational structure of L_3, and
  (ii) for every element x of L_2 holds $\downarrow\!\!x$ is directed and non empty.

  If L_2 satisfies axiom of approximation, then L_3 satisfies axiom of approximation.

Let T be a continuous non empty poset. One can verify that every topological augmentation of T is continuous.

The following propositions are true:

(19) Let T, S be topological spaces, R be a refinement of T and S, and W be a subset of R. If $W \in$ the topology of T or $W \in$ the topology of S, then W is open.

(20) Let T, S be topological spaces, R be a refinement of T and S, V be a subset of T, and W be a subset of R. If $W = V$, then if V is open, then W is open.

(21) Let T, S be topological spaces. Suppose the carrier of T = the carrier of S. Let R be a refinement of T and S, V be a subset of T, and W be a subset of R. If $W = V$, then if V is closed, then W is closed.

(22) Let T be a non empty topological space and K, O be sets such that $K \subseteq O$ and $O \subseteq$ the topology of T. Then

(i) if K is a basis of T, then O is a basis of T, and

(ii) if K is a prebasis of T, then O is a prebasis of T.

(23) Let T_1, T_2 be non empty topological spaces. Suppose the carrier of T_1 = the carrier of T_2. Let T be a refinement of T_1 and T_2, B_2 be a prebasis of T_1, and B_3 be a prebasis of T_2. Then $B_2 \cup B_3$ is a prebasis of T.

(24) Let T_1, S_1, T_2, S_2 be non empty topological spaces, R_1 be a refinement of T_1 and S_1, R_2 be a refinement of T_2 and S_2, f be a map from T_1 into T_2, g be a map from S_1 into S_2, and h be a map from R_1 into R_2. Suppose $h = f$ and $h = g$. If f is continuous and g is continuous, then h is continuous.

(25) Let T be a non empty topological space, K be a prebasis of T, N be a net in T, and p be a point of T. Suppose that for every subset A of T such that $p \in A$ and $A \in K$ holds N is eventually in A. Then $p \in \operatorname{Lim} N$.

(26) Let T be a non empty topological space, N be a net in T, and S be a subset of T. If N is eventually in S, then $\operatorname{Lim} N \subseteq \overline{S}$.

(27) Let R be a non empty relational structure and X be a non empty subset of R. Then the mapping of $\langle X; \mathrm{id}\rangle = \mathrm{id}_X$ and the mapping of $\langle X^{\mathrm{op}}; \mathrm{id}\rangle = \mathrm{id}_X$.

(28) For every reflexive antisymmetric non empty relational structure R and for every element x of R holds $\uparrow x \cap \downarrow x = \{x\}$.

3. Lawson Topology

Let T be a reflexive non empty FR-structure. We say that T is Lawson if and only if:

(Def. 3) $\omega(T) \cup \sigma(T)$ is a prebasis of T.

Next we state the proposition

(29) Let R be a complete lattice, L_1 be a lower correct topological augmentation of R, S be a Scott topological augmentation of R, and T be a correct topological augmentation of R. Then T is Lawson if and only if T is a refinement of S and L_1.

Let R be a complete lattice. One can check that there exists a topological augmentation of R which is Lawson, strict, and correct.

Let us observe that there exists a top-lattice which is Scott, complete, and strict and there exists a complete strict top-lattice which is Lawson and continuous.

We now state three propositions:

(30) For every Lawson complete top-lattice T holds $\sigma(T) \cup \{-\uparrow x : x$ ranges over elements of $T\}$ is a prebasis of T.

(31) Let T be a Lawson complete top-lattice. Then $\sigma(T) \cup \{W \setminus \uparrow x; W$ ranges over subsets of T, x ranges over elements of T: $W \in \sigma(T)\}$ is a prebasis of T.

(32) Let T be a Lawson complete top-lattice. Then $\{W \setminus \uparrow F; W$ ranges over subsets of T, F ranges over subsets of T: $W \in \sigma(T) \wedge F$ is finite$\}$ is a basis of T.

Let T be a complete lattice. The functor $\lambda(T)$ yields a family of subsets of T and is defined as follows:

(Def. 4) For every Lawson correct topological augmentation S of T holds $\lambda(T) =$ the topology of S.

We now state a number of propositions:

(33) For every complete lattice R holds $\lambda(R) = \mathrm{UniCl}(\mathrm{FinMeetCl}(\sigma(R) \cup \omega(R)))$.

(34) Let R be a complete lattice, T be a lower correct topological augmentation of R, S be a Scott correct topological augmentation of R, and M be a refinement of S and T. Then $\lambda(R) =$ the topology of M.

(35) For every lower up-complete top-lattice T and for every subset A of T such that A is open holds A has the property (S).

(36) For every Lawson complete top-lattice T and for every subset A of T such that A is open holds A has the property (S).

(37) Let S be a Scott complete top-lattice, T be a Lawson correct topological augmentation of S, and A be a subset of S. If A is open, then for every subset C of T such that $C = A$ holds C is open.

(38) Let T be a Lawson complete top-lattice and x be an element of T. Then $\uparrow x$ is closed and $\downarrow x$ is closed and $\{x\}$ is closed.

(39) For every Lawson complete top-lattice T and for every element x of T holds $-\uparrow x$ is open and $-\downarrow x$ is open and $-\{x\}$ is open.

(40) For every Lawson complete continuous top-lattice T and for every element x of T holds $\twoheaduparrow x$ is open and $-\twoheaduparrow x$ is closed.

(41) Let S be a Scott complete top-lattice, T be a Lawson correct topological augmentation of S, and A be an upper subset of T. If A is open, then for every subset C of S such that $C = A$ holds C is open.

(42) Let T be a Lawson complete top-lattice and A be a lower subset of T.

Then A is closed if and only if A is closed under directed sups.

(43) For every Lawson complete top-lattice T and for every non empty filtered subset F of T holds $\mathrm{Lim}\langle F^{\mathrm{op}}; \mathrm{id}\rangle = \{\inf F\}$.

Let us observe that every complete top-lattice which is Lawson is also T_1 and compact.

Let us observe that every complete continuous top-lattice which is Lawson is also Hausdorff.

References

[1] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(**5**):719–725, 1991.
[2] Grzegorz Bancerek. Bounds in posets and relational substructures. *Formalized Mathematics*, 6(**1**):81–91, 1997.
[3] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(**1**):93–107, 1997.
[4] Grzegorz Bancerek. The "way-below" relation. *Formalized Mathematics*, 6(**1**):169–176, 1997.
[5] Grzegorz Bancerek. Bases and refinements of topologies. *Formalized Mathematics*, 7(**1**):35–43, 1998.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[9] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, Berlin, Heidelberg, New York, 1980.
[10] Artur Korniłowicz. Cartesian products of relations and relational structures. *Formalized Mathematics*, 6(**1**):145–152, 1997.
[11] Artur Korniłowicz. Meet–continuous lattices. *Formalized Mathematics*, 6(**1**):159–167, 1997.
[12] Artur Korniłowicz. On the topological properties of meet-continuous lattices. *Formalized Mathematics*, 6(**2**):269–277, 1997.
[13] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[14] Alexander Yu. Shibakov and Andrzej Trybulec. The Cantor set. *Formalized Mathematics*, 5(**2**):233–236, 1996.
[15] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[16] Andrzej Trybulec. Moore-Smith convergence. *Formalized Mathematics*, 6(**2**):213–225, 1997.
[17] Andrzej Trybulec. Scott topology. *Formalized Mathematics*, 6(**2**):311–319, 1997.
[18] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[19] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[20] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

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