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Some Properties of Special Polygonal Curves

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Summary. In the paper some auxiliary theorems are proved, needed in the proof of the second part of the Jordan curve theorem for special polygons. They deal mostly with characteristic points of plane non empty compacts introduced in [5], operation *mid* introduced in [19] and the predicate "f is in the area of g" (f and g : finite sequences of points of the plane) introduced in [28].

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The notation and terminology used here are introduced in the following papers: [21], [32], [6], [22], [24], [7], [2], [3], [30], [4], [27], [15], [16], [20], [26], [19], [9], [18], [11], [12], [13], [1], [23], [5], [10], [14], [17], [29], [28], [31], [25], [8], and [33].

1. Preliminaries

In this paper i, j, k, n are natural numbers.

The following propositions are true:

(1) For all sets A, B, C such that A misses B holds $A \cap (B \cup C) = A \cap C$.

(2) For all sets A, B, C, p such that $A \subseteq B$ and $B \cap C = \{p\}$ and $p \in A$ holds $A \cap C = \{p\}$.

(3) For all real numbers q, r, s, t such that $t \geqslant 0$ and $t \leqslant 1$ and $s = (1-t) \cdot q + t \cdot r$ and $q \leqslant s$ and $r < s$ holds $t = 0$.

(4) For all real numbers q, r, s, t such that $t \geqslant 0$ and $t \leqslant 1$ and $s = (1-t) \cdot q + t \cdot r$ and $q \geqslant s$ and $r > s$ holds $t = 0$.

(5) If $i -' k \leqslant j$, then $i \leqslant j + k$.

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(6) If $i \leqslant j + k$, then $i -' k \leqslant j$.

(7) If $i \leqslant j -' k$ and $k \leqslant j$, then $i + k \leqslant j$.

(8) If $j + k \leqslant i$, then $k \leqslant i -' j$.

(9) If $k \leqslant i$ and $i < j$, then $i -' k < j -' k$.

(10) If $i < j$ and $k < j$, then $i -' k < j -' k$.

(11) Let D be a non empty set, f be a non empty finite sequence of elements of D, and g be a finite sequence of elements of D. Then $\pi_{\mathrm{len}(g \frown f)}(g \frown f) = \pi_{\mathrm{len}\, f} f$.

(12) For all sets a, b, c, d holds the indices of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \{\langle 1, 1\rangle, \langle 1, 2\rangle, \langle 2, 1\rangle, \langle 2, 2\rangle\}$.

2. Euclidean Space

We now state four propositions:

(13) For all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ and for every real number r such that $0 < r$ and $p = (1 - r) \cdot p + r \cdot q$ holds $p = q$.

(14) For all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ and for every real number r such that $r < 1$ and $p = (1 - r) \cdot q + r \cdot p$ holds $p = q$.

(15) For all points p, q of $\mathcal{E}^n_{\mathrm{T}}$ such that $p = \frac{1}{2} \cdot (p + q)$ holds $p = q$.

(16) For all points p, q, r of $\mathcal{E}^n_{\mathrm{T}}$ such that $q \in \mathcal{L}(p, r)$ and $r \in \mathcal{L}(p, q)$ holds $q = r$.

3. Euclidean Plane

One can prove the following propositions:

(17) Let A be a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, p be an element of the carrier of $\mathcal{E}^2$, and r be a real number. If $A = \mathrm{Ball}(p, r)$, then A is connected.

(18) For all subsets A, B of $\mathcal{E}^2_{\mathrm{T}}$ such that A is open and B is a component of A holds B is open.

(19) For all points p, q, r of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathcal{L}(p, q)$ is horizontal and $r \in \mathcal{L}(p, q)$ holds $p_{\mathbf{2}} = r_{\mathbf{2}}$.

(20) For all points p, q, r of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathcal{L}(p, q)$ is vertical and $r \in \mathcal{L}(p, q)$ holds $p_{\mathbf{1}} = r_{\mathbf{1}}$.

(21) For all points p, q, r, s of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathcal{L}(p, q)$ is horizontal and $\mathcal{L}(r, s)$ is horizontal and $\mathcal{L}(p, q)$ meets $\mathcal{L}(r, s)$ holds $p_{\mathbf{2}} = r_{\mathbf{2}}$.

(22) For all points p, q, r of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathcal{L}(p,q)$ is vertical and $\mathcal{L}(q,r)$ is horizontal holds $\mathcal{L}(p,q) \cap \mathcal{L}(q,r) = \{q\}$.

(23) For all points p, q, r, s of $\mathcal{E}^2_{\mathrm{T}}$ such that $\mathcal{L}(p,q)$ is horizontal and $\mathcal{L}(s,r)$ is vertical and $r \in \mathcal{L}(p,q)$ holds $\mathcal{L}(p,q) \cap \mathcal{L}(s,r) = \{r\}$.

4. Miscellaneous

In the sequel p, q denote points of $\mathcal{E}^2_{\mathrm{T}}$ and G denotes a Go-board.

Next we state two propositions:

(24) If $1 \leqslant j$ and $j \leqslant k$ and $k \leqslant \operatorname{width} G$ and $1 \leqslant i$ and $i \leqslant \operatorname{len} G$, then $(G_{i,j})_{\mathbf{2}} \leqslant (G_{i,k})_{\mathbf{2}}$.

(25) If $1 \leqslant j$ and $j \leqslant \operatorname{width} G$ and $1 \leqslant i$ and $i \leqslant k$ and $k \leqslant \operatorname{len} G$, then $(G_{i,j})_{\mathbf{1}} \leqslant (G_{k,j})_{\mathbf{1}}$.

In the sequel C denotes a subset of $\mathcal{E}^2_{\mathrm{T}}$.

We now state a number of propositions:

(26) $\mathcal{L}(\text{NW-corner}\, C, \text{NE-corner}\, C) \subseteq \widetilde{\mathcal{L}}(\text{SpStSeq}\, C)$.

(27) $\text{N-most}\, C \subseteq \mathcal{L}(\text{NW-corner}\, C, \text{NE-corner}\, C)$.

(28) For every non empty compact subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\text{N-min}\, C \in \mathcal{L}(\text{NW-corner}\, C, \text{NE-corner}\, C)$.

(29) $\mathcal{L}(\text{NW-corner}\, C, \text{NE-corner}\, C)$ is horizontal.

(30) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and i, j be natural numbers. Suppose f is a special sequence and $1 \leqslant i$ and $i \leqslant j$ and $j \leqslant \operatorname{len} f$. Then LE $\pi_i f$, $\pi_j f$, $\widetilde{\mathcal{L}}(f)$, $\pi_1 f$, $\pi_{\operatorname{len} f} f$.

(31) Let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $\pi_1 g \neq p$ and $(\pi_1 g)_{\mathbf{1}} = p_{\mathbf{1}}$ or $(\pi_1 g)_{\mathbf{2}} = p_{\mathbf{2}}$ and g is a special sequence and $\mathcal{L}(p, \pi_1 g) \cap \widetilde{\mathcal{L}}(g) = \{\pi_1 g\}$. Then $\langle p \rangle \frown g$ is a special sequence.

(32) Let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $\pi_{\operatorname{len} g} g \neq p$ and $(\pi_{\operatorname{len} g} g)_{\mathbf{1}} = p_{\mathbf{1}}$ or $(\pi_{\operatorname{len} g} g)_{\mathbf{2}} = p_{\mathbf{2}}$ and g is a special sequence and $\mathcal{L}(p, \pi_{\operatorname{len} g} g) \cap \widetilde{\mathcal{L}}(g) = \{\pi_{\operatorname{len} g} g\}$. Then $g \frown \langle p \rangle$ is a special sequence.

(33) Let f be a S-sequence in $\mathbb{R}^2$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $1 < j$ and $j \leqslant \operatorname{len} f$ and $p \in \widetilde{\mathcal{L}}(\operatorname{mid}(f, 1, j))$, then LE p, $\pi_j f$, $\widetilde{\mathcal{L}}(f)$, $\pi_1 f$, $\pi_{\operatorname{len} f} f$.

(34) For every finite sequence h of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that $i \in \operatorname{dom} h$ and $j \in \operatorname{dom} h$ holds $\widetilde{\mathcal{L}}(\operatorname{mid}(h, i, j)) \subseteq \widetilde{\mathcal{L}}(h)$.

(35) If $1 \leqslant i$ and $i < j$, then for every finite sequence f of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that $j \leqslant \operatorname{len} f$ holds $\widetilde{\mathcal{L}}(\operatorname{mid}(f, i, j)) = \mathcal{L}(f, i) \cup \widetilde{\mathcal{L}}(\operatorname{mid}(f, i+1, j))$.

(36) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. If $1 \leqslant i$, then if $i < j$ and $j \leqslant \operatorname{len} f$, then $\widetilde{\mathcal{L}}(\operatorname{mid}(f, i, j)) = \widetilde{\mathcal{L}}(\operatorname{mid}(f, i, j -' 1)) \cup \mathcal{L}(f, j -' 1)$.

(37) Let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose g is a special sequence and $p_{\mathbf{1}} = (\pi_1 g)_{\mathbf{1}}$ or $p_{\mathbf{2}} = (\pi_1 g)_{\mathbf{2}}$ and $\mathcal{L}(p, \pi_1 g) \cap \widetilde{\mathcal{L}}(g) = \{\pi_1 g\}$ and $p \neq \pi_1 g$. Then $\langle p \rangle \frown g$ is a special sequence.

(38) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) f is a special sequence,

(ii) g is a special sequence,

(iii) $(\pi_{\mathrm{len}\, f} f)_{\mathbf{1}} = (\pi_1 g)_{\mathbf{1}}$ or $(\pi_{\mathrm{len}\, f} f)_{\mathbf{2}} = (\pi_1 g)_{\mathbf{2}}$,

(iv) $\widetilde{\mathcal{L}}(f)$ misses $\widetilde{\mathcal{L}}(g)$,

(v) $\mathcal{L}(\pi_{\mathrm{len}\, f} f, \pi_1 g) \cap \widetilde{\mathcal{L}}(f) = \{\pi_{\mathrm{len}\, f} f\}$, and

(vi) $\mathcal{L}(\pi_{\mathrm{len}\, f} f, \pi_1 g) \cap \widetilde{\mathcal{L}}(g) = \{\pi_1 g\}$.

Then $f \frown g$ is a special sequence.

(39) For every S-sequence f in $\mathbb{R}^2$ and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in \widetilde{\mathcal{L}}(f)$ holds $\pi_1 \,\lfloor\, f, p = \pi_1 f$.

(40) Let f be a S-sequence in $\mathbb{R}^2$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If $1 \leqslant j$ and $j < \mathrm{len}\, f$ and $p \in \mathcal{L}(f, j)$ and $q \in \mathcal{L}(\pi_j f, p)$, then LE q, p, $\widetilde{\mathcal{L}}(f)$, $\pi_1 f$, $\pi_{\mathrm{len}\, f} f$.

5. Special Circular Sequences

Next we state the proposition

(41) For every non constant standard special circular sequence f holds LeftComp(f) is open and RightComp(f) is open.

Let f be a non constant standard special circular sequence. One can verify the following observations:

* $\widetilde{\mathcal{L}}(f)$ is non vertical and non horizontal,
* LeftComp(f) is region, and
* RightComp(f) is region.

One can prove the following propositions:

(42) For every non constant standard special circular sequence f holds RightComp(f) misses $\widetilde{\mathcal{L}}(f)$.

(43) For every non constant standard special circular sequence f holds LeftComp(f) misses $\widetilde{\mathcal{L}}(f)$.

(44) For every non constant standard special circular sequence f holds $\mathrm{i}_{\mathrm{WN}}\, f < \mathrm{i}_{\mathrm{EN}}\, f$.

(45) Let f be a non constant standard special circular sequence. Then there exists i such that $1 \leqslant i$ and $i < \mathrm{len}$ the Go-board of f and N-min $\widetilde{\mathcal{L}}(f) =$ (the Go-board of $f)_{i,\mathrm{width\,the\ Go\text{-}board\ of}\ f}$.

(46) Let f be a clockwise oriented non constant standard special circular sequence. Suppose $i \in \operatorname{dom}$ the Go-board of f and $\pi_1 f =$ (the Go-board of $f)_{i,\mathrm{width\,the\,Go\text{-}board\,of}\,f}$ and $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$. Then $\pi_2 f =$ (the Go-board of $f)_{i+1,\mathrm{width\,the\,Go\text{-}board\,of}\,f}$ and $\pi_{\operatorname{len} f -' 1} f =$ (the Go-board of $f)_{i,\mathrm{width\,the\,Go\text{-}board\,of}\,f -' 1}$.

(47) Let f be a non constant standard special circular sequence. If $1 \leqslant i$ and $i < j$ and $j \leqslant \operatorname{len} f$ and $\pi_1 f \in \widetilde{\mathcal{L}}(\operatorname{mid}(f,i,j))$, then $i = 1$ or $j = \operatorname{len} f$.

(48) Let f be a clockwise oriented non constant standard special circular sequence. If $\pi_1 f = \text{N-min}\,\widetilde{\mathcal{L}}(f)$, then $\mathcal{L}(\pi_1 f, \pi_2 f) \subseteq \widetilde{\mathcal{L}}(\operatorname{SpStSeq}\widetilde{\mathcal{L}}(f))$.

6. Rectangular Sequences

We now state the proposition

(49) Let f be a rectangular finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$, then $p_{\mathbf{1}} = \text{W-bound}\,\widetilde{\mathcal{L}}(f)$ or $p_{\mathbf{1}} = \text{E-bound}\,\widetilde{\mathcal{L}}(f)$ or $p_{\mathbf{2}} = \text{S-bound}\,\widetilde{\mathcal{L}}(f)$ or $p_{\mathbf{2}} = \text{N-bound}\,\widetilde{\mathcal{L}}(f)$.

One can check that there exists a special circular sequence which is rectangular.

The following propositions are true:

(50) Let f be a rectangular special circular sequence and g be a S-sequence in $\mathbb{R}^2$. If $\pi_1 g \in \operatorname{LeftComp}(f)$ and $\pi_{\operatorname{len} g} g \in \operatorname{RightComp}(f)$, then $\widetilde{\mathcal{L}}(f)$ meets $\widetilde{\mathcal{L}}(g)$.

(51) For every rectangular special circular sequence f holds $\operatorname{SpStSeq}\widetilde{\mathcal{L}}(f) = f$.

(52) Let f be a rectangular special circular sequence. Then $\widetilde{\mathcal{L}}(f) = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{1}} = \text{W-bound}\,\widetilde{\mathcal{L}}(f) \wedge p_{\mathbf{2}} \leqslant \text{N-bound}\,\widetilde{\mathcal{L}}(f) \wedge p_{\mathbf{2}} \geqslant \text{S-bound}\,\widetilde{\mathcal{L}}(f) \vee p_{\mathbf{1}} \leqslant \text{E-bound}\,\widetilde{\mathcal{L}}(f) \wedge p_{\mathbf{1}} \geqslant \text{W-bound}\,\widetilde{\mathcal{L}}(f) \wedge p_{\mathbf{2}} = \text{N-bound}\,\widetilde{\mathcal{L}}(f) \vee p_{\mathbf{1}} \leqslant \text{E-bound}\,\widetilde{\mathcal{L}}(f) \wedge p_{\mathbf{1}} \geqslant \text{W-bound}\,\widetilde{\mathcal{L}}(f) \wedge p_{\mathbf{2}} = \text{S-bound}\,\widetilde{\mathcal{L}}(f) \vee p_{\mathbf{1}} = \text{E-bound}\,\widetilde{\mathcal{L}}(f) \wedge p_{\mathbf{2}} \leqslant \text{N-bound}\,\widetilde{\mathcal{L}}(f) \wedge p_{\mathbf{2}} \geqslant \text{S-bound}\,\widetilde{\mathcal{L}}(f)\}$.

(53) For every rectangular special circular sequence f holds the Go-board of $f = \begin{pmatrix} \pi_4 f & \pi_1 f \\ \pi_3 f & \pi_2 f \end{pmatrix}$.

(54) Let f be a rectangular special circular sequence. Then $\operatorname{LeftComp}(f) = \{p : \text{W-bound}\,\widetilde{\mathcal{L}}(f) \not\leqslant p_{\mathbf{1}} \vee p_{\mathbf{1}} \not\leqslant \text{E-bound}\,\widetilde{\mathcal{L}}(f) \vee \text{S-bound}\,\widetilde{\mathcal{L}}(f) \not\leqslant p_{\mathbf{2}} \vee p_{\mathbf{2}} \not\leqslant \text{N-bound}\,\widetilde{\mathcal{L}}(f)\}$ and $\operatorname{RightComp}(f) = \{q : \text{W-bound}\,\widetilde{\mathcal{L}}(f) < q_{\mathbf{1}} \wedge q_{\mathbf{1}} < \text{E-bound}\,\widetilde{\mathcal{L}}(f) \wedge \text{S-bound}\,\widetilde{\mathcal{L}}(f) < q_{\mathbf{2}} \wedge q_{\mathbf{2}} < \text{N-bound}\,\widetilde{\mathcal{L}}(f)\}$.

One can check that there exists a rectangular special circular sequence which is clockwise oriented.

One can check that every rectangular special circular sequence is clockwise oriented.

Next we state four propositions:

(55) Let f be a rectangular special circular sequence and g be a S-sequence in $\mathbb{R}^2$. If $\pi_1 g \in \mathrm{LeftComp}(f)$ and $\pi_{\mathrm{len}\, g} g \in \mathrm{RightComp}(f)$, then $\mathrm{LPoint}(\widetilde{\mathcal{L}}(g), \pi_1 g, \pi_{\mathrm{len}\, g} g, \widetilde{\mathcal{L}}(f)) \neq \text{NW-corner}\, \widetilde{\mathcal{L}}(f)$.

(56) Let f be a rectangular special circular sequence and g be a S-sequence in $\mathbb{R}^2$. If $\pi_1 g \in \mathrm{LeftComp}(f)$ and $\pi_{\mathrm{len}\, g} g \in \mathrm{RightComp}(f)$, then $\mathrm{LPoint}(\widetilde{\mathcal{L}}(g), \pi_1 g, \pi_{\mathrm{len}\, g} g, \widetilde{\mathcal{L}}(f)) \neq \text{SE-corner}\, \widetilde{\mathcal{L}}(f)$.

(57) Let f be a rectangular special circular sequence and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If $\text{W-bound}\, \widetilde{\mathcal{L}}(f) > p_{\mathbf{1}}$ or $p_{\mathbf{1}} > \text{E-bound}\, \widetilde{\mathcal{L}}(f)$ or $\text{S-bound}\, \widetilde{\mathcal{L}}(f) > p_{\mathbf{2}}$ or $p_{\mathbf{2}} > \text{N-bound}\, \widetilde{\mathcal{L}}(f)$, then $p \in \mathrm{LeftComp}(f)$.

(58) For every clockwise oriented non constant standard special circular sequence f such that $\pi_1 f = \text{N-min}\, \widetilde{\mathcal{L}}(f)$ holds $\mathrm{LeftComp}(\mathrm{SpStSeq}\, \widetilde{\mathcal{L}}(f)) \subseteq \mathrm{LeftComp}(f)$.

7. In the Area

Next we state a number of propositions:

(59) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. Then $\langle p, q\rangle$ is in the area of f if and only if $\langle p\rangle$ is in the area of f and $\langle q\rangle$ is in the area of f.

(60) Let f be a rectangular finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose $\langle p\rangle$ is in the area of f but $p_{\mathbf{1}} = \text{W-bound}\, \widetilde{\mathcal{L}}(f)$ or $p_{\mathbf{1}} = \text{E-bound}\, \widetilde{\mathcal{L}}(f)$ or $p_{\mathbf{2}} = \text{S-bound}\, \widetilde{\mathcal{L}}(f)$ or $p_{\mathbf{2}} = \text{N-bound}\, \widetilde{\mathcal{L}}(f)$. Then $p \in \widetilde{\mathcal{L}}(f)$.

(61) Let f be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$, p, q be points of $\mathcal{E}^2_{\mathrm{T}}$, and r be a real number. Suppose $0 \leqslant r$ and $r \leqslant 1$ and $\langle p, q\rangle$ is in the area of f. Then $\langle (1-r)\cdot p + r \cdot q\rangle$ is in the area of f.

(62) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$. If g is in the area of f and $i \in \mathrm{dom}\, g$, then $\langle \pi_i g\rangle$ is in the area of f.

(63) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. If g is in the area of f and $p \in \widetilde{\mathcal{L}}(g)$, then $\langle p\rangle$ is in the area of f.

(64) Let f be a rectangular finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If $q \notin \widetilde{\mathcal{L}}(f)$ and $\langle p, q\rangle$ is in the area of f, then $\mathcal{L}(p,q) \cap \widetilde{\mathcal{L}}(f) \subseteq \{p\}$.

(65) Let f be a rectangular finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If $p \in \widetilde{\mathcal{L}}(f)$ and $q \notin \widetilde{\mathcal{L}}(f)$ and $\langle q\rangle$ is in the area of f, then $\mathcal{L}(p,q) \cap \widetilde{\mathcal{L}}(f) = \{p\}$.

(66) Let f be a non constant standard special circular sequence. Suppose $1 \leqslant i$ and $i \leqslant \operatorname{len}$ the Go-board of f and $1 \leqslant j$ and $j \leqslant \operatorname{width}$ the Go-board of f. Then $\langle(\text{the Go-board of } f)_{i,j}\rangle$ is in the area of f.

(67) Let g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If $\langle p, q\rangle$ is in the area of g, then $\langle \frac{1}{2} \cdot (p+q)\rangle$ is in the area of g.

(68) For all finite sequences f, g of elements of $\mathcal{E}^2_{\mathrm{T}}$ such that g is in the area of f holds $\operatorname{Rev}(g)$ is in the area of f.

(69) Let f, g be finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2_{\mathrm{T}}$. Suppose that

(i) g is in the area of f,

(ii) $\langle p\rangle$ is in the area of f,

(iii) g is a special sequence, and

(iv) there exists a natural number i such that $1 \leqslant i$ and $i+1 \leqslant \operatorname{len} g$ and $p \in \mathcal{L}(g, i)$.

Then $\downharpoonleft g, p$ is in the area of f.

(70) Let f be a non constant standard special circular sequence and g be a finite sequence of elements of $\mathcal{E}^2_{\mathrm{T}}$. Then g is in the area of f if and only if g is in the area of $\operatorname{SpStSeq} \widetilde{\mathcal{L}}(f)$.

(71) Let f be a rectangular special circular sequence and g be a S-sequence in $\mathbb{R}^2$. If $\pi_1 g \in \operatorname{LeftComp}(f)$ and $\pi_{\operatorname{len} g} g \in \operatorname{RightComp}(f)$, then $\downharpoonright \operatorname{LPoint}(\widetilde{\mathcal{L}}(g), \pi_1 g, \pi_{\operatorname{len} g} g, \widetilde{\mathcal{L}}(f)), g$ is in the area of f.

(72) Let f be a non constant standard special circular sequence. Suppose $1 \leqslant i$ and $i < \operatorname{len}$ the Go-board of f and $1 \leqslant j$ and $j < \operatorname{width}$ the Go-board of f. Then $\operatorname{Int} \operatorname{cell}(\text{the Go-board of } f, i, j)$ misses $\widetilde{\mathcal{L}}(\operatorname{SpStSeq} \widetilde{\mathcal{L}}(f))$.

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