FORMALIZED MATHEMATICS
Volume 7, Number 1, 1998
University of Białystok

On the Characterization of Modular and Distributive Lattices[1]

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Summary. This article contains definitions of the "pentagon" lattice N_5 and the "diamond" lattice M_3. It is followed by the characterization of modular and distributive lattices depending on the possible shape of substructures. The last part treats of interval-like sublattices of any lattice.

MML Identifier: `YELLOW11`.

The papers [8], [5], [1], [7], [6], [3], [4], and [2] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(1) $3 = \{0, 1, 2\}$.

(2) $2 \setminus 1 = \{1\}$.

(3) $3 \setminus 1 = \{1, 2\}$.

(4) $3 \setminus 2 = \{2\}$.

(5) Let L be an antisymmetric reflexive relational structure with g.l.b.'s and l.u.b.'s and a, b be elements of L. Then $a \sqcap b = b$ if and only if $a \sqcup b = a$.

(6) For every lattice L and for all elements a, b, c of L holds $(a \sqcap b) \sqcup (a \sqcap c) \leqslant a \sqcap (b \sqcup c)$.

(7) For every lattice L and for all elements a, b, c of L holds $a \sqcup (b \sqcap c) \leqslant (a \sqcup b) \sqcap (a \sqcup c)$.

[1]This work has been supported by KBN Grant 8 T11C 018 12.

ISSN 1426–2630

(8) For every lattice L and for all elements a, b, c of L such that $a \leqslant c$ holds $a \sqcup (b \sqcap c) \leqslant (a \sqcup b) \sqcap c$.

2. Diamond and Pentagon

The relational structure N_5 is defined as follows:

(Def. 1) $N_5 = \langle \{0, 3 \setminus 1, 2, 3 \setminus 2, 3\}, \subseteq \rangle$.

Let us note that N_5 is strict reflexive transitive and antisymmetric and N_5 has g.l.b.'s and l.u.b.'s.

The relational structure M_3 is defined by:

(Def. 2) $M_3 = \langle \{0, 1, 2 \setminus 1, 3 \setminus 2, 3\}, \subseteq \rangle$.

Let us note that M_3 is strict reflexive transitive and antisymmetric and M_3 has g.l.b.'s and l.u.b.'s.

One can prove the following two propositions:

(9) Let L be a lattice. Then the following statements are equivalent
 (i) there exists a full sublattice K of L such that N_5 and K are isomorphic,
 (ii) there exist elements a, b, c, d, e of L such that $a \neq b$ and $a \neq c$ and $a \neq d$ and $a \neq e$ and $b \neq c$ and $b \neq d$ and $b \neq e$ and $c \neq d$ and $c \neq e$ and $d \neq e$ and $a \sqcap b = a$ and $a \sqcap c = a$ and $c \sqcap e = c$ and $d \sqcap e = d$ and $b \sqcap c = a$ and $b \sqcap d = b$ and $c \sqcap d = a$ and $b \sqcup c = e$ and $c \sqcup d = e$.

(10) Let L be a lattice. Then the following statements are equivalent
 (i) there exists a full sublattice K of L such that M_3 and K are isomorphic,
 (ii) there exist elements a, b, c, d, e of L such that $a \neq b$ and $a \neq c$ and $a \neq d$ and $a \neq e$ and $b \neq c$ and $b \neq d$ and $b \neq e$ and $c \neq d$ and $c \neq e$ and $d \neq e$ and $a \sqcap b = a$ and $a \sqcap c = a$ and $a \sqcap d = a$ and $b \sqcap e = b$ and $c \sqcap e = c$ and $d \sqcap e = d$ and $b \sqcap c = a$ and $b \sqcap d = a$ and $c \sqcap d = a$ and $b \sqcup c = e$ and $b \sqcup d = e$ and $c \sqcup d = e$.

Let L be a non empty relational structure. We say that L is modular if and only if:

(Def. 3) For all elements a, b, c of L such that $a \leqslant c$ holds $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap c$.

Let us note that every non empty antisymmetric reflexive relational structure with g.l.b.'s which is distributive is also modular.

Next we state two propositions:

(11) Let L be a lattice. Then L is modular if and only if it is not true that there exists a full sublattice K of L such that N_5 and K are isomorphic.

(12) Let L be a lattice. Suppose L is modular. Then L is distributive if and only if it is not true that there exists a full sublattice K of L such that M_3 and K are isomorphic.

3. Intervals of a Lattice

Let L be a non empty relational structure and let a, b be elements of L. The functor $[a, b]$ yielding a subset of L is defined as follows:

(Def. 4) For every element c of L holds $c \in [a, b]$ iff $a \leqslant c$ and $c \leqslant b$.

Let L be a non empty relational structure and let I_1 be a subset of L. We say that I_1 is interval if and only if:

(Def. 5) There exist elements a, b of L such that $I_1 = [a, b]$.

Let L be a non empty reflexive transitive relational structure. One can check that every subset of L which is non empty and interval is also directed and every subset of L which is non empty and interval is also filtered.

Let L be a non empty relational structure and let a, b be elements of L. Observe that $[a, b]$ is interval.

Next we state the proposition

(13) For every non empty reflexive transitive relational structure L and for all elements a, b of L holds $[a, b] = \uparrow a \cap \downarrow b$.

Let L be a poset with g.l.b.'s and let a, b be elements of L. Observe that $\mathrm{sub}([a, b])$ is meet-inheriting.

Let L be a poset with l.u.b.'s and let a, b be elements of L. Note that $\mathrm{sub}([a, b])$ is join-inheriting.

One can prove the following proposition

(14) Let L be a lattice and a, b be elements of L. If L is modular, then $\mathrm{sub}([b, a \sqcup b])$ and $\mathrm{sub}([a \sqcap b, a])$ are isomorphic.

Let us mention that there exists a lattice which is finite and non empty.

Let us note that every semilattice which is finite is also lower-bounded.

Let us note that every lattice which is finite is also complete.

References

[1] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(**5**):719–725, 1991.
[2] Grzegorz Bancerek. Bounds in posets and relational substructures. *Formalized Mathematics*, 6(**1**):81–91, 1997.
[3] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(**1**):93–107, 1997.
[4] Czesław Byliński. Galois connections. *Formalized Mathematics*, 6(**1**):131–143, 1997.
[5] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[6] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[7] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[8] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

Received April 3, 1998