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Completely-Irreducible Elements[1]

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Summary. The article is a translation of [5, 92–93].

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The terminology and notation used here are introduced in the following articles: [16], [1], [14], [12], [15], [13], [3], [4], [9], [6], [10], [11], [2], [7], and [8].

1. Preliminaries

The following propositions are true:

(1) For every sup-semilattice L and for all elements x, y of L holds $\bigsqcap_L(\uparrow x \cap \uparrow y) = x \sqcup y$.

(2) For every semilattice L and for all elements x, y of L holds $\bigsqcup_L(\downarrow x \cap \downarrow y) = x \sqcap y$.

(3) Let L be a non empty relational structure and x, y be elements of L. If x is maximal in (the carrier of L) $\setminus \uparrow y$, then $\uparrow x \setminus \{x\} = \uparrow x \cap \uparrow y$.

(4) Let L be a non empty relational structure and x, y be elements of L. If x is minimal in (the carrier of L) $\setminus \downarrow y$, then $\downarrow x \setminus \{x\} = \downarrow x \cap \downarrow y$.

(5) Let L be a poset with l.u.b.'s, X, Y be subsets of L, and X', Y' be subsets of $L^{\rm op}$. If $X = X'$ and $Y = Y'$, then $X \sqcup Y = X' \sqcap Y'$.

(6) Let L be a poset with g.l.b.'s, X, Y be subsets of L, and X', Y' be subsets of $L^{\rm op}$. If $X = X'$ and $Y = Y'$, then $X \sqcap Y = X' \sqcup Y'$.

(7) For every non empty reflexive transitive relational structure L holds $\mathrm{Filt}(L) = \mathrm{Ids}(L^{\rm op})$.

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(8) For every non empty reflexive transitive relational structure L holds $\mathrm{Ids}(L) = \mathrm{Filt}(L^{\mathrm{op}})$.

2. Free Generation Set

Let S, T be complete non empty posets. A map from S into T is said to be a CLHomomorphism of S, T if:

(Def. 1) It is directed-sups-preserving and infs-preserving.

Let S be a continuous complete non empty poset and let A be a subset of S. We say that A is a free generator set if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let T be a continuous complete non empty poset and f be a function from A into the carrier of T. Then there exists a CLHomomorphism h of S, T such that $h \restriction A = f$ and for every CLHomomorphism h' of S, T such that $h' \restriction A = f$ holds $h' = h$.

Let L be an upper-bounded non empty poset. One can check that $\mathrm{Filt}(L)$ is non empty.

The following propositions are true:

(9) For every set X and for every non empty subset Y of $\langle \mathrm{Filt}(2^X_\subseteq), \subseteq \rangle$ holds $\bigcap Y$ is a filter of $2^X_\subseteq$.

(10) For every set X and for every non empty subset Y of $\langle \mathrm{Filt}(2^X_\subseteq), \subseteq \rangle$ holds $\inf Y$ exists in $\langle \mathrm{Filt}(2^X_\subseteq), \subseteq \rangle$ and $\bigsqcap_{(\langle \mathrm{Filt}(2^X_\subseteq), \subseteq \rangle)} Y = \bigcap Y$.

(11) For every set X holds 2^X is a filter of $2^X_\subseteq$.

(12) For every set X holds $\{X\}$ is a filter of $2^X_\subseteq$.

(13) For every set X holds $\langle \mathrm{Filt}(2^X_\subseteq), \subseteq \rangle$ is upper-bounded.

(14) For every set X holds $\langle \mathrm{Filt}(2^X_\subseteq), \subseteq \rangle$ is lower-bounded.

(15) For every set X holds $\top_{\langle \mathrm{Filt}(2^X_\subseteq), \subseteq \rangle} = 2^X$.

(16) For every set X holds $\bot_{\langle \mathrm{Filt}(2^X_\subseteq), \subseteq \rangle} = \{X\}$.

(17) For every non empty set X and for every non empty subset Y of $\langle X, \subseteq \rangle$ such that $\sup Y$ exists in $\langle X, \subseteq \rangle$ holds $\bigcup Y \subseteq \sup Y$.

(18) For every upper-bounded semilattice L holds $\langle \mathrm{Filt}(L), \subseteq \rangle$ is complete.

Let L be an upper-bounded semilattice. Note that $\langle \mathrm{Filt}(L), \subseteq \rangle$ is complete.

3. Completely-Irreducible Elements

Let L be a non empty relational structure and let p be an element of L. We say that p is completely-irreducible if and only if:

(Def. 3) Min $\uparrow p \setminus \{p\}$ exists in L.

We now state the proposition

(19) Let L be a non empty relational structure and p be an element of L. If p is completely-irreducible, then $\bigsqcap_L(\uparrow p \setminus \{p\}) \neq p$.

Let L be a non empty relational structure. The functor $\mathrm{Irr}\, L$ yielding a subset of L is defined by:

(Def. 4) For every element x of L holds $x \in \mathrm{Irr}\, L$ iff x is completely-irreducible.

The following propositions are true:

(20) Let L be a non empty poset and p be an element of L. Then p is completely-irreducible if and only if there exists an element q of L such that $p < q$ and for every element s of L such that $p < s$ holds $q \leqslant s$ and $\uparrow p = \{p\} \cup \uparrow q$.

(21) For every upper-bounded non empty poset L holds $\top_L \notin \mathrm{Irr}\, L$.

(22) For every semilattice L holds $\mathrm{Irr}\, L \subseteq \mathrm{IRR}(L)$.

(23) For every semilattice L and for every element x of L such that x is completely-irreducible holds x is irreducible.

(24) Let L be a non empty poset and x be an element of L. Suppose x is completely-irreducible. Let X be a subset of L. If $\inf X$ exists in L and $x = \inf X$, then $x \in X$.

(25) For every non empty poset L and for every subset X of L such that X is order-generating holds $\mathrm{Irr}\, L \subseteq X$.

(26) Let L be a complete lattice and p be an element of L. Given an element k of L such that p is maximal in (the carrier of $L) \setminus \uparrow k$. Then p is completely-irreducible.

(27) Let L be a transitive antisymmetric relational structure with l.u.b.'s and p, q, u be elements of L. Suppose $p < q$ and for every element s of L such that $p < s$ holds $q \leqslant s$ and $u \not\leqslant p$. Then $p \sqcup u = q \sqcup u$.

(28) Let L be a distributive lattice and p, q, u be elements of L. Suppose $p < q$ and for every element s of L such that $p < s$ holds $q \leqslant s$ and $u \not\leqslant p$. Then $u \sqcap q \not\leqslant p$.

(29) Let L be a distributive complete lattice. Suppose L^{op} is meet-continuous. Let p be an element of L. Suppose p is completely-irreducible. Then (the carrier of $L) \setminus \downarrow p$ is an open filter of L.

(30) Let L be a distributive complete lattice. Suppose L^{op} is meet-continuous. Let p be an element of L. Suppose p is completely-irreducible. Then there exists an element k of L such that $k \in$ the carrier of $\mathrm{CompactSublatt}(L)$ and p is maximal in (the carrier of $L) \setminus \uparrow k$.

(31) Let L be a lower-bounded algebraic lattice and x, y be elements of L. Suppose $y \not\leqslant x$. Then there exists an element p of L such that p is

completely-irreducible and $x \leqslant p$ and $y \nleqslant p$.

(32) Let L be a lower-bounded algebraic lattice. Then $\mathrm{Irr}\, L$ is order-generating and for every subset X of L such that X is order-generating holds $\mathrm{Irr}\, L \subseteq X$.

(33) For every lower-bounded algebraic lattice L and for every element s of L holds $s = \bigsqcap_L (\uparrow s \cap \mathrm{Irr}\, L)$.

(34) Let L be a complete non empty poset, X be a subset of L, and p be an element of L. If p is completely-irreducible and $p = \inf X$, then $p \in X$.

(35) Let L be a complete algebraic lattice and p be an element of L. Suppose p is completely-irreducible. Then $p = \bigsqcap_L \{x; x$ ranges over elements of L: $x \in \uparrow p \ \wedge \ \bigvee_{k\,:\,\text{element of } L}$ ($k \in$ the carrier of CompactSublatt(L) $\wedge$ x is maximal in (the carrier of L) $\setminus \uparrow k$)$\}$.

(36) Let L be a complete algebraic lattice and p be an element of L. Then there exists an element k of L such that $k \in$ the carrier of CompactSublatt(L) and p is maximal in (the carrier of L) $\setminus \uparrow k$ if and only if p is completely-irreducible.

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