FORMALIZED MATHEMATICS
Volume 7, Number 1, 1998
University of Białystok

The Field of Quotients Over an Integral Domain

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Summary. We introduce the field of quotients over an integral domain following the well-known construction using pairs over integral domains. In addition we define ring homomorphisms and prove some basic facts about fields of quotients including their universal property.

MML Identifier: `QUOFIELD`.

The papers [1], [13], [10], [2], [3], [7], [9], [11], [12], [5], [6], [8], and [4] provide the terminology and notation for this paper.

1. Preliminaries

Let I be a non empty zero structure. The functor $\mathrm{Q}(I)$ is a subset of $[\!:$ the carrier of I, the carrier of $I\,:\!]$ and is defined by:

(Def. 1) For every set u holds $u \in \mathrm{Q}(I)$ iff there exist elements a, b of the carrier of I such that $u = \langle a, b\rangle$ and $b \neq 0_I$.

Next we state the proposition

(1) For every non degenerated non empty multiplicative loop with zero structure I holds $\mathrm{Q}(I)$ is non empty.

The following two propositions are true:

(2) Let I be a non degenerated non empty multiplicative loop with zero structure and u be an element of $\mathrm{Q}(I)$. Then $u_{\mathbf{2}} \neq 0_I$.

(3) Let I be a non degenerated non empty multiplicative loop with zero structure and u be an element of $\mathrm{Q}(I)$. Then $u_{\mathbf{1}}$ is an element of the carrier of I and $u_{\mathbf{2}}$ is an element of the carrier of I.

ISSN 1426–2630

Let I be a non degenerated integral domain-like non empty double loop structure and let u, v be elements of $\mathrm{Q}(I)$. The functor $u+v$ yielding an element of $\mathrm{Q}(I)$ is defined by:

(Def. 2) $u+v = \langle u_{\mathbf{1}} \cdot v_{\mathbf{2}} + v_{\mathbf{1}} \cdot u_{\mathbf{2}},\ u_{\mathbf{2}} \cdot v_{\mathbf{2}} \rangle$.

Let I be a non degenerated integral domain-like non empty double loop structure and let u, v be elements of $\mathrm{Q}(I)$. The functor $u \cdot v$ yielding an element of $\mathrm{Q}(I)$ is defined as follows:

(Def. 3) $u \cdot v = \langle u_{\mathbf{1}} \cdot v_{\mathbf{1}},\ u_{\mathbf{2}} \cdot v_{\mathbf{2}} \rangle$.

The following two propositions are true:

(4) Let I be a non degenerated integral domain-like associative commutative Abelian add-associative distributive non empty double loop structure and u, v, w be elements of $\mathrm{Q}(I)$. Then $u+(v+w) = (u+v)+w$ and $u+v = v+u$.

(5) Let I be a non degenerated integral domain-like associative commutative Abelian non empty double loop structure and u, v, w be elements of $\mathrm{Q}(I)$. Then $u \cdot (v \cdot w) = (u \cdot v) \cdot w$ and $u \cdot v = v \cdot u$.

Let I be a non degenerated integral domain-like associative commutative Abelian add-associative distributive non empty double loop structure and let u, v be elements of $\mathrm{Q}(I)$. Let us notice that the functor $u+v$ is commutative.

Let I be a non degenerated integral domain-like associative commutative Abelian non empty double loop structure and let u, v be elements of $\mathrm{Q}(I)$. Let us note that the functor $u \cdot v$ is commutative.

Let I be a non degenerated non empty multiplicative loop with zero structure and let u be an element of $\mathrm{Q}(I)$. The functor $\mathrm{QClass}(u)$ is a subset of $\mathrm{Q}(I)$ and is defined as follows:

(Def. 4) For every element z of $\mathrm{Q}(I)$ holds $z \in \mathrm{QClass}(u)$ iff $z_{\mathbf{1}} \cdot u_{\mathbf{2}} = z_{\mathbf{2}} \cdot u_{\mathbf{1}}$.

The following proposition is true

(6) Let I be a non degenerated commutative non empty multiplicative loop with zero structure and u be an element of $\mathrm{Q}(I)$. Then $u \in \mathrm{QClass}(u)$.

Let I be a non degenerated commutative non empty multiplicative loop with zero structure and let u be an element of $\mathrm{Q}(I)$. Observe that $\mathrm{QClass}(u)$ is non empty.

Let I be a non degenerated non empty multiplicative loop with zero structure. The functor $\mathrm{Quot}(I)$ is a family of subsets of $\mathrm{Q}(I)$ and is defined by:

(Def. 5) For every subset A of $\mathrm{Q}(I)$ holds $A \in \mathrm{Quot}(I)$ iff there exists an element u of $\mathrm{Q}(I)$ such that $A = \mathrm{QClass}(u)$.

Next we state the proposition

(7) For every non degenerated non empty multiplicative loop with zero structure I holds $\mathrm{Quot}(I)$ is non empty.

Next we state two propositions:

(8) Let I be a non degenerated integral domain-like ring and u, v be elements of $\mathrm{Q}(I)$. If there exists an element w of $\mathrm{Quot}(I)$ such that $u \in w$ and $v \in w$, then $u_{\mathbf{1}} \cdot v_{\mathbf{2}} = v_{\mathbf{1}} \cdot u_{\mathbf{2}}$.

(9) For every non degenerated integral domain-like ring I and for all elements u, v of $\mathrm{Quot}(I)$ such that $u \cap v \neq \emptyset$ holds $u = v$.

2. Defining the Operations

Let I be a non degenerated integral domain-like ring and let u, v be elements of $\mathrm{Quot}(I)$. The functor $u +_{\mathrm{q}} v$ yielding an element of $\mathrm{Quot}(I)$ is defined by the condition (Def. 6).

(Def. 6) Let z be an element of $\mathrm{Q}(I)$. Then $z \in u +_{\mathrm{q}} v$ if and only if there exist elements a, b of $\mathrm{Q}(I)$ such that $a \in u$ and $b \in v$ and $z_{\mathbf{1}} \cdot (a_{\mathbf{2}} \cdot b_{\mathbf{2}}) = z_{\mathbf{2}} \cdot (a_{\mathbf{1}} \cdot b_{\mathbf{2}} + b_{\mathbf{1}} \cdot a_{\mathbf{2}})$.

Let I be a non degenerated integral domain-like ring and let u, v be elements of $\mathrm{Quot}(I)$. The functor $u \cdot_{\mathrm{q}} v$ yielding an element of $\mathrm{Quot}(I)$ is defined by the condition (Def. 7).

(Def. 7) Let z be an element of $\mathrm{Q}(I)$. Then $z \in u \cdot_{\mathrm{q}} v$ if and only if there exist elements a, b of $\mathrm{Q}(I)$ such that $a \in u$ and $b \in v$ and $z_{\mathbf{1}} \cdot (a_{\mathbf{2}} \cdot b_{\mathbf{2}}) = z_{\mathbf{2}} \cdot (a_{\mathbf{1}} \cdot b_{\mathbf{1}})$.

Next we state the proposition

(10) Let I be a non degenerated non empty multiplicative loop with zero structure and u be an element of $\mathrm{Q}(I)$. Then $\mathrm{QClass}(u)$ is an element of $\mathrm{Quot}(I)$.

We now state two propositions:

(11) For every non degenerated integral domain-like ring I and for all elements u, v of $\mathrm{Q}(I)$ holds $\mathrm{QClass}(u) +_{\mathrm{q}} \mathrm{QClass}(v) = \mathrm{QClass}(u + v)$.

(12) For every non degenerated integral domain-like ring I and for all elements u, v of $\mathrm{Q}(I)$ holds $\mathrm{QClass}(u) \cdot_{\mathrm{q}} \mathrm{QClass}(v) = \mathrm{QClass}(u \cdot v)$.

Let I be a non degenerated integral domain-like ring. The functor $0_{\mathrm{q}}(I)$ yielding an element of $\mathrm{Quot}(I)$ is defined by:

(Def. 8) For every element z of $\mathrm{Q}(I)$ holds $z \in 0_{\mathrm{q}}(I)$ iff $z_{\mathbf{1}} = 0_I$.

Let I be a non degenerated integral domain-like ring. The functor $1_{\mathrm{q}}(I)$ yielding an element of $\mathrm{Quot}(I)$ is defined as follows:

(Def. 9) For every element z of $\mathrm{Q}(I)$ holds $z \in 1_{\mathrm{q}}(I)$ iff $z_{\mathbf{1}} = z_{\mathbf{2}}$.

Let I be a non degenerated integral domain-like ring and let u be an element of $\mathrm{Quot}(I)$. The functor $-_{\mathrm{q}}u$ yielding an element of $\mathrm{Quot}(I)$ is defined by:

(Def. 10) For every element z of $\mathrm{Q}(I)$ holds $z \in -_{\mathrm{q}}u$ iff there exists an element a of $\mathrm{Q}(I)$ such that $a \in u$ and $z_{\mathbf{1}} \cdot a_{\mathbf{2}} = z_{\mathbf{2}} \cdot -a_{\mathbf{1}}$.

Let I be a non degenerated integral domain-like ring and let u be an element of $\mathrm{Quot}(I)$. Let us assume that $u \neq 0_{\mathrm{q}}(I)$. The functor u_{q}^{-1} yields an element of $\mathrm{Quot}(I)$ and is defined by:

(Def. 11) For every element z of $\mathrm{Q}(I)$ holds $z \in u_{\mathrm{q}}^{-1}$ iff there exists an element a of $\mathrm{Q}(I)$ such that $a \in u$ and $z_{\mathbf{1}} \cdot a_{\mathbf{1}} = z_{\mathbf{2}} \cdot a_{\mathbf{2}}$.

The following propositions are true:

(13) Let I be a non degenerated integral domain-like ring and u, v, w be elements of $\mathrm{Quot}(I)$. Then $u +_{\mathrm{q}} (v +_{\mathrm{q}} w) = (u +_{\mathrm{q}} v) +_{\mathrm{q}} w$ and $u +_{\mathrm{q}} v = v +_{\mathrm{q}} u$.

(14) For every non degenerated integral domain-like ring I and for every element u of $\mathrm{Quot}(I)$ holds $u +_{\mathrm{q}} 0_{\mathrm{q}}(I) = u$ and $0_{\mathrm{q}}(I) +_{\mathrm{q}} u = u$.

(15) Let I be a non degenerated integral domain-like ring and u, v, w be elements of $\mathrm{Quot}(I)$. Then $u \cdot_{\mathrm{q}} (v \cdot_{\mathrm{q}} w) = (u \cdot_{\mathrm{q}} v) \cdot_{\mathrm{q}} w$ and $u \cdot_{\mathrm{q}} v = v \cdot_{\mathrm{q}} u$.

(16) For every non degenerated integral domain-like ring I and for every element u of $\mathrm{Quot}(I)$ holds $u \cdot_{\mathrm{q}} 1_{\mathrm{q}}(I) = u$ and $1_{\mathrm{q}}(I) \cdot_{\mathrm{q}} u = u$.

(17) For every non degenerated integral domain-like ring I and for all elements u, v, w of $\mathrm{Quot}(I)$ holds $(u +_{\mathrm{q}} v) \cdot_{\mathrm{q}} w = (u \cdot_{\mathrm{q}} w) +_{\mathrm{q}} (v \cdot_{\mathrm{q}} w)$.

(18) For every non degenerated integral domain-like ring I and for all elements u, v, w of $\mathrm{Quot}(I)$ holds $u \cdot_{\mathrm{q}} (v +_{\mathrm{q}} w) = (u \cdot_{\mathrm{q}} v) +_{\mathrm{q}} (u \cdot_{\mathrm{q}} w)$.

(19) For every non degenerated integral domain-like ring I and for every element u of $\mathrm{Quot}(I)$ holds $u +_{\mathrm{q}} -_{\mathrm{q}}u = 0_{\mathrm{q}}(I)$ and $-_{\mathrm{q}}u +_{\mathrm{q}} u = 0_{\mathrm{q}}(I)$.

(20) Let I be a non degenerated integral domain-like ring and u be an element of $\mathrm{Quot}(I)$. If $u \neq 0_{\mathrm{q}}(I)$, then $u \cdot_{\mathrm{q}} u_{\mathrm{q}}^{-1} = 1_{\mathrm{q}}(I)$ and $u_{\mathrm{q}}^{-1} \cdot_{\mathrm{q}} u = 1_{\mathrm{q}}(I)$.

(21) For every non degenerated integral domain-like ring I holds $1_{\mathrm{q}}(I) \neq 0_{\mathrm{q}}(I)$.

Let I be a non degenerated integral domain-like ring. The functor $+_{\mathrm{q}}(I)$ yielding a binary operation on $\mathrm{Quot}(I)$ is defined as follows:

(Def. 12) For all elements u, v of $\mathrm{Quot}(I)$ holds $(+_{\mathrm{q}}(I))(u, v) = u +_{\mathrm{q}} v$.

Let I be a non degenerated integral domain-like ring. The functor $\cdot_{\mathrm{q}}(I)$ yields a binary operation on $\mathrm{Quot}(I)$ and is defined as follows:

(Def. 13) For all elements u, v of $\mathrm{Quot}(I)$ holds $(\cdot_{\mathrm{q}}(I))(u, v) = u \cdot_{\mathrm{q}} v$.

Let I be a non degenerated integral domain-like ring. The functor $-_{\mathrm{q}}(I)$ yields a unary operation on $\mathrm{Quot}(I)$ and is defined as follows:

(Def. 14) For every element u of $\mathrm{Quot}(I)$ holds $(-_{\mathrm{q}}(I))(u) = -_{\mathrm{q}}u$.

Let I be a non degenerated integral domain-like ring. The functor ${}_{\mathrm{q}}^{-1}(I)$ yields a unary operation on $\mathrm{Quot}(I)$ and is defined as follows:

(Def. 15) For every element u of $\mathrm{Quot}(I)$ holds $({}_{\mathrm{q}}^{-1}(I))(u) = u_{\mathrm{q}}^{-1}$.

We now state a number of propositions:

(22) For every non degenerated integral domain-like ring I and for all elements u, v, w of $\mathrm{Quot}(I)$ holds $(+_{\mathrm{q}}(I))((+_{\mathrm{q}}(I))(u, v), w) = (+_{\mathrm{q}}(I))(u, (+_{\mathrm{q}}(I))(v, w))$.

(23) For every non degenerated integral domain-like ring I and for all elements u, v of $\mathrm{Quot}(I)$ holds $(+_{\mathrm{q}}(I))(u, v) = (+_{\mathrm{q}}(I))(v, u)$.

(24) For every non degenerated integral domain-like ring I and for every element u of $\mathrm{Quot}(I)$ holds $(+_{\mathrm{q}}(I))(u, 0_{\mathrm{q}}(I)) = u$ and $(+_{\mathrm{q}}(I))(0_{\mathrm{q}}(I), u) = u$.

(25) For every non degenerated integral domain-like ring I and for all elements u, v, w of $\mathrm{Quot}(I)$ holds $(\cdot_{\mathrm{q}}(I))((\cdot_{\mathrm{q}}(I))(u, v), w) = (\cdot_{\mathrm{q}}(I))(u, (\cdot_{\mathrm{q}}(I))(v, w))$.

(26) For every non degenerated integral domain-like ring I and for all elements u, v of $\mathrm{Quot}(I)$ holds $(\cdot_{\mathrm{q}}(I))(u, v) = (\cdot_{\mathrm{q}}(I))(v, u)$.

(27) For every non degenerated integral domain-like ring I and for every element u of $\mathrm{Quot}(I)$ holds $(\cdot_{\mathrm{q}}(I))(u, 1_{\mathrm{q}}(I)) = u$ and $(\cdot_{\mathrm{q}}(I))(1_{\mathrm{q}}(I), u) = u$.

(28) Let I be a non degenerated integral domain-like ring and u, v, w be elements of $\mathrm{Quot}(I)$. Then $(\cdot_{\mathrm{q}}(I))((+_{\mathrm{q}}(I))(u, v), w) = (+_{\mathrm{q}}(I))((\cdot_{\mathrm{q}}(I))(u, w), (\cdot_{\mathrm{q}}(I))(v, w))$.

(29) Let I be a non degenerated integral domain-like ring and u, v, w be elements of $\mathrm{Quot}(I)$. Then $(\cdot_{\mathrm{q}}(I))(u, (+_{\mathrm{q}}(I))(v, w)) = (+_{\mathrm{q}}(I))((\cdot_{\mathrm{q}}(I))(u, v), (\cdot_{\mathrm{q}}(I))(u, w))$.

(30) Let I be a non degenerated integral domain-like ring and u be an element of $\mathrm{Quot}(I)$. Then $(+_{\mathrm{q}}(I))(u, (-_{\mathrm{q}}(I))(u)) = 0_{\mathrm{q}}(I)$ and $(+_{\mathrm{q}}(I))((-_{\mathrm{q}}(I))(u), u) = 0_{\mathrm{q}}(I)$.

(31) Let I be a non degenerated integral domain-like ring and u be an element of $\mathrm{Quot}(I)$. If $u \neq 0_{\mathrm{q}}(I)$, then $(\cdot_{\mathrm{q}}(I))(u, ({}_{\mathrm{q}}^{-1}(I))(u)) = 1_{\mathrm{q}}(I)$ and $(\cdot_{\mathrm{q}}(I))(({}_{\mathrm{q}}^{-1}(I))(u), u) = 1_{\mathrm{q}}(I)$.

3. Defining the Field of Quotients

Let I be a non degenerated integral domain-like ring. The field of quotients of I yields a strict double loop structure and is defined as follows:

(Def. 16) The field of quotients of $I = \langle \mathrm{Quot}(I), +_{\mathrm{q}}(I), \cdot_{\mathrm{q}}(I), 1_{\mathrm{q}}(I), 0_{\mathrm{q}}(I) \rangle$.

Let I be a non degenerated integral domain-like ring. Observe that the field of quotients of I is non empty.

The following propositions are true:

(32) Let I be a non degenerated integral domain-like ring. Then

(i) the carrier of the field of quotients of $I = \mathrm{Quot}(I)$,

(ii) the addition of the field of quotients of $I = +_{\mathrm{q}}(I)$,

(iii) the multiplication of the field of quotients of $I = \cdot_{\rm q}(I)$,
(iv) the zero of the field of quotients of $I = 0_{\rm q}(I)$, and
(v) the unity of the field of quotients of $I = 1_{\rm q}(I)$.

(33) Let I be a non degenerated integral domain-like ring and u, v be elements of the carrier of the field of quotients of I. Then $(+_{\rm q}(I))(u, v)$ is an element of the carrier of the field of quotients of I.

(34) Let I be a non degenerated integral domain-like ring and u be an element of the carrier of the field of quotients of I. Then $(-_{\rm q}(I))(u)$ is an element of the carrier of the field of quotients of I.

(35) Let I be a non degenerated integral domain-like ring and u, v be elements of the carrier of the field of quotients of I. Then $(\cdot_{\rm q}(I))(u, v)$ is an element of the carrier of the field of quotients of I.

(36) Let I be a non degenerated integral domain-like ring and u be an element of the carrier of the field of quotients of I. Then $({}_{\rm q}^{-1}(I))(u)$ is an element of the carrier of the field of quotients of I.

(37) Let I be a non degenerated integral domain-like ring and u, v be elements of the carrier of the field of quotients of I. Then $u + v = (+_{\rm q}(I))(u, v)$.

Let I be a non degenerated integral domain-like ring. One can verify that the field of quotients of I is add-associative right zeroed and right complementable.

Next we state a number of propositions:

(38) Let I be a non degenerated integral domain-like ring and u be an element of the carrier of the field of quotients of I. Then $-u = (-_{\rm q}(I))(u)$.

(39) Let I be a non degenerated integral domain-like ring and u, v be elements of the carrier of the field of quotients of I. Then $u \cdot v = (\cdot_{\rm q}(I))(u, v)$.

(40) Let I be a non degenerated integral domain-like ring. Then $1_{\text{the field of quotients of } I} = 1_{\rm q}(I)$ and $0_{\text{the field of quotients of } I} = 0_{\rm q}(I)$.

(41) Let I be a non degenerated integral domain-like ring and u, v, w be elements of the carrier of the field of quotients of I. Then $(u + v) + w = u + (v + w)$.

(42) Let I be a non degenerated integral domain-like ring and u, v be elements of the carrier of the field of quotients of I. Then $u + v = v + u$.

(43) Let I be a non degenerated integral domain-like ring and u be an element of the carrier of the field of quotients of I. Then $u + 0_{\text{the field of quotients of } I} = u$.

(44) Let I be a non degenerated integral domain-like ring and u be an element of the carrier of the field of quotients of I. Then $u + -u = 0_{\text{the field of quotients of } I}$.

(45) Let I be a non degenerated integral domain-like ring and u be an element of the carrier of the field of quotients of I. Then $1_{\text{the field of quotients of } I} \cdot u = u$.

(46) Let I be a non degenerated integral domain-like ring and u, v be elements of the carrier of the field of quotients of I. Then $u \cdot v = v \cdot u$.

(47) Let I be a non degenerated integral domain-like ring and u, v, w be elements of the carrier of the field of quotients of I. Then $(u \cdot v) \cdot w = u \cdot (v \cdot w)$.

(48) Let I be a non degenerated integral domain-like ring and u be an element of the carrier of the field of quotients of I. Suppose $u \neq 0_{\text{the field of quotients of } I}$. Then there exists an element v of the carrier of the field of quotients of I such that $u \cdot v = 1_{\text{the field of quotients of } I}$.

(49) Let I be a non degenerated integral domain-like ring. Then the field of quotients of I is an add-associative right zeroed right complementable Abelian commutative associative left unital distributive field-like non degenerated non empty double loop structure.

Let I be a non degenerated integral domain-like ring. Note that the field of quotients of I is Abelian commutative associative left unital distributive field-like and non degenerated.

Next we state the proposition

(50) Let I be a non degenerated integral domain-like ring and x be an element of the carrier of the field of quotients of I. Suppose $x \neq 0_{\text{the field of quotients of } I}$. Let a be an element of the carrier of I. Suppose $a \neq 0_I$. Let u be an element of $\mathrm{Q}(I)$. Suppose $x = \mathrm{QClass}(u)$ and $u = \langle a, 1_I \rangle$. Let v be an element of $\mathrm{Q}(I)$. If $v = \langle 1_I, a \rangle$, then $x^{-1} = \mathrm{QClass}(v)$.

Let us observe that every add-associative right zeroed right complementable commutative associative left unital distributive field-like non degenerated non empty double loop structure is integral domain-like and right unital.

One can check that there exists a non empty double loop structure which is add-associative, right zeroed, right complementable, Abelian, commutative, associative, left unital, distributive, field-like, and non degenerated.

Let F be a commutative associative left unital distributive field-like non empty double loop structure and let x, y be elements of the carrier of F. The functor $\frac{x}{y}$ yields an element of the carrier of F and is defined as follows:

(Def. 17) $\frac{x}{y} = x \cdot y^{-1}$.

One can prove the following propositions:

(51) Let F be a non degenerated field-like ring and a, b, c, d be elements of the carrier of F. If $b \neq 0_F$ and $d \neq 0_F$, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$.

(52) Let F be a non degenerated field-like ring and a, b, c, d be elements of the carrier of F. If $b \neq 0_F$ and $d \neq 0_F$, then $\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + c \cdot b}{b \cdot d}$.

4. Defining Ring Homomorphisms

Let R, S be non empty double loop structures and let f be a map from R into S. We say that f is a ring homomorphism if and only if:

(Def. 21)[1] f is additive, multiplicative, and unity-preserving.

Let R, S be non empty double loop structures. One can verify that every map from R into S which is ring homomorphism is also additive, multiplicative, and unity-preserving and every map from R into S which is additive, multiplicative, and unity-preserving is also a ring homomorphism.

Let R, S be non empty double loop structures and let f be a map from R into S. We say that f is a ring epimorphism if and only if:

(Def. 22) f is a ring homomorphism and $\operatorname{rng} f =$ the carrier of S.

We say that f is a ring monomorphism if and only if:

(Def. 23) f is a ring homomorphism and one-to-one.

We introduce f is an embedding as a synonym of f is a ring monomorphism.

Let R, S be non empty double loop structures and let f be a map from R into S. We say that f is a ring isomorphism if and only if:

(Def. 24) f is a ring monomorphism and a ring epimorphism.

Let R, S be non empty double loop structures. Note that every map from R into S which is ring isomorphism is also a ring monomorphism and a ring epimorphism and every map from R into S which is ring monomorphism and ring epimorphism is also a ring isomorphism.

We now state several propositions:

(53) For all rings R, S and for every map f from R into S such that f is a ring homomorphism holds $f(0_R) = 0_S$.

(54) Let R, S be rings and f be a map from R into S. Suppose f is a ring monomorphism. Let x be an element of the carrier of R. Then $f(x) = 0_S$ if and only if $x = 0_R$.

(55) Let R, S be non degenerated field-like rings and f be a map from R into S. Suppose f is a ring homomorphism. Let x be an element of the carrier of R. If $x \neq 0_R$, then $f(x^{-1}) = f(x)^{-1}$.

(56) Let R, S be non degenerated field-like rings and f be a map from R into S. Suppose f is a ring homomorphism. Let x, y be elements of the carrier of R. If $y \neq 0_R$, then $f(x \cdot y^{-1}) = f(x) \cdot f(y)^{-1}$.

(57) Let R, S, T be rings and f be a map from R into S. Suppose f is a ring homomorphism. Let g be a map from S into T. If g is a ring homomorphism, then $g \cdot f$ is a ring homomorphism.

[1]The definitions (Def. 18)–(Def. 20) have been removed.

(58) For every non empty double loop structure R holds id_R is a ring homomorphism.

Let R, S be non empty double loop structures. We say that R is embedded in S if and only if:

(Def. 25) There exists a map from R into S which is a ring monomorphism.

Let R, S be non empty double loop structures. We say that R is ring isomorphic to S if and only if:

(Def. 26) There exists a map from R into S which is a ring isomorphism.

Let us note that the predicate R is ring isomorphic to S is symmetric.

5. Some Further Properties

Let I be a non empty zero structure and let x, y be elements of the carrier of I. Let us assume that $y \neq 0_I$. The functor $\mathrm{quotient}(x, y)$ yielding an element of $\mathrm{Q}(I)$ is defined as follows:

(Def. 27) $\mathrm{quotient}(x, y) = \langle x, y \rangle$.

Let I be a non degenerated integral domain-like ring. The canonical homomorphism of I into quotient field is a map from I into the field of quotients of I and is defined by the condition (Def. 28).

(Def. 28) Let x be an element of the carrier of I. Then (the canonical homomorphism of I into quotient field)$(x) = \mathrm{QClass}(\mathrm{quotient}(x, 1_I))$.

Next we state four propositions:

(59) Let I be a non degenerated integral domain-like ring. Then the canonical homomorphism of I into quotient field is a ring homomorphism.

(60) Let I be a non degenerated integral domain-like ring. Then the canonical homomorphism of I into quotient field is an embedding.

(61) For every non degenerated integral domain-like ring I holds I is embedded in the field of quotients of I.

(62) Let F be a non degenerated field-like integral domain-like ring. Then F is ring isomorphic to the field of quotients of F.

Let I be a non degenerated integral domain-like ring. Note that the field of quotients of I is integral domain-like right unital and right-distributive.

One can prove the following proposition

(63) Let I be a non degenerated integral domain-like ring. Then the field of quotients of the field of quotients of I is ring isomorphic to the field of quotients of I.

Let I be a non empty double loop structure, let F be a non empty double loop structure, and let f be a map from I into F. We say that F is a field of quotients for I via f if and only if the conditions (Def. 29) are satisfied.

(Def. 29)(i) f is a ring monomorphism, and

(ii) for every add-associative right zeroed right complementable Abelian commutative associative left unital distributive field-like non degenerated non empty double loop structure F' and for every map f' from I into F' such that f' is a ring monomorphism there exists a map h from F into F' such that h is a ring homomorphism and $h \cdot f = f'$ and for every map h' from F into F' such that h' is a ring homomorphism and $h' \cdot f = f'$ holds $h' = h$.

Next we state two propositions:

(64) Let I be a non degenerated integral domain-like ring. Then there exists an add-associative right zeroed right complementable Abelian commutative associative left unital distributive field-like non degenerated non empty double loop structure F and there exists a map f from I into F such that F is a field of quotients for I via f.

(65) Let I be an integral domain-like ring, F, F' be add-associative right zeroed right complementable Abelian commutative associative left unital distributive field-like non degenerated non empty double loop structures, f be a map from I into F, and f' be a map from I into F'. Suppose F is a field of quotients for I via f and F' is a field of quotients for I via f'. Then F is ring isomorphic to F'.

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Received May 4, 1998