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The Product of the Families of the Groups

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The terminology and notation used here are introduced in the following articles: [6], [1], [4], [2], [3], [9], [10], [8], [12], [13], [11], [7], and [5].

1. Preliminaries

In this paper a, b, c, d, e, f are sets.

Next we state three propositions:

(1) If $\langle a \rangle = \langle b \rangle$, then $a = b$.

(2) If $\langle a, b \rangle = \langle c, d \rangle$, then $a = c$ and $b = d$.

(3) If $\langle a, b, c \rangle = \langle d, e, f \rangle$, then $a = d$ and $b = e$ and $c = f$.

2. The Product of the Families of the Groups

We use the following convention: i, I denote sets, f, g, h denote functions, and s denotes a many sorted set indexed by I.

Let R be a binary relation. We say that R is semigroup yielding if and only if:

(Def. 1) For every set y such that $y \in \operatorname{rng} R$ holds y is a non empty semigroup.

Let us note that every function which is semigroup yielding is also 1-sorted yielding.

Let I be a set. One can verify that there exists a many sorted set indexed by I which is semigroup yielding.

Let us observe that there exists a function which is semigroup yielding.

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Let I be a set. A family of semigroups indexed by I is a semigroup yielding many sorted set indexed by I.

Let I be a non empty set, let F be a family of semigroups indexed by I, and let i be an element of I. Then $F(i)$ is a non empty semigroup.

Let I be a set and let F be a family of semigroups indexed by I. One can verify that the support of F is non-empty.

Let I be a set and let F be a family of semigroups indexed by I. The functor $\prod F$ yielding a strict semigroup is defined by the conditions (Def. 2).

(Def. 2)(i) The carrier of $\prod F = \prod$ (the support of F), and

(ii) for all elements f, g of $\prod$ (the support of F) and for every set i such that $i \in I$ there exists a non empty semigroup F_1 and there exists a function h such that $F_1 = F(i)$ and $h =$ (the multiplication of $\prod F)(f, g)$ and $h(i) =$ (the multiplication of $F_1)(f(i), g(i))$.

Let I be a set and let F be a family of semigroups indexed by I. Note that $\prod F$ is non empty.

Let I be a set and let F be a family of semigroups indexed by I. Observe that every element of the carrier of $\prod F$ is function-like and relation-like.

Let I be a set, let F be a family of semigroups indexed by I, and let f, g be elements of $\prod$ (the support of F). Observe that (the multiplication of $\prod F)(f, g)$ is function-like and relation-like.

One can prove the following proposition

(4) Let F be a family of semigroups indexed by I, G be a non empty semigroup, p, q be elements of the carrier of $\prod F$, and x, y be elements of the carrier of G. Suppose $i \in I$ and $G = F(i)$ and $f = p$ and $g = q$ and $h = p \cdot q$ and $f(i) = x$ and $g(i) = y$. Then $x \cdot y = h(i)$.

Let I be a set and let F be a family of semigroups indexed by I. We say that F is group-like if and only if:

(Def. 3) For every set i such that $i \in I$ there exists a group-like non empty semigroup F_1 such that $F_1 = F(i)$.

We say that F is associative if and only if:

(Def. 4) For every set i such that $i \in I$ there exists an associative non empty semigroup F_1 such that $F_1 = F(i)$.

We say that F is commutative if and only if:

(Def. 5) For every set i such that $i \in I$ there exists a commutative non empty semigroup F_1 such that $F_1 = F(i)$.

Let I be a non empty set and let F be a family of semigroups indexed by I. Let us observe that F is group-like if and only if:

(Def. 6) For every element i of I holds $F(i)$ is group-like.

Let us observe that F is associative if and only if:

(Def. 7) For every element i of I holds $F(i)$ is associative.

Let us observe that F is commutative if and only if:

(Def. 8) For every element i of I holds $F(i)$ is commutative.

Let I be a set. Note that there exists a family of semigroups indexed by I which is group-like, associative, and commutative.

Let I be a set and let F be a group-like family of semigroups indexed by I. Note that $\prod F$ is group-like.

Let I be a set and let F be an associative family of semigroups indexed by I. One can check that $\prod F$ is associative.

Let I be a set and let F be a commutative family of semigroups indexed by I. One can verify that $\prod F$ is commutative.

We now state several propositions:

(5) Let F be a family of semigroups indexed by I and G be a non empty semigroup. If $i \in I$ and $G = F(i)$ and $\prod F$ is group-like, then G is group-like.

(6) Let F be a family of semigroups indexed by I and G be a non empty semigroup. If $i \in I$ and $G = F(i)$ and $\prod F$ is associative, then G is associative.

(7) Let F be a family of semigroups indexed by I and G be a non empty semigroup. If $i \in I$ and $G = F(i)$ and $\prod F$ is commutative, then G is commutative.

(8) Let F be a group-like family of semigroups indexed by I. Suppose that for every set i such that $i \in I$ there exists a group-like non empty semigroup G such that $G = F(i)$ and $s(i) = 1_G$. Then $s = 1_{\prod F}$.

(9) Let F be a group-like family of semigroups indexed by I and G be a group-like non empty semigroup. If $i \in I$ and $G = F(i)$ and $f = 1_{\prod F}$, then $f(i) = 1_G$.

(10) Let F be an associative group-like family of semigroups indexed by I and x be an element of the carrier of $\prod F$. Suppose that

(i) $x = g$, and

(ii) for every set i such that $i \in I$ there exists a group G and there exists an element y of the carrier of G such that $G = F(i)$ and $s(i) = y^{-1}$ and $y = g(i)$.

Then $s = x^{-1}$.

(11) Let F be an associative group-like family of semigroups indexed by I, x be an element of the carrier of $\prod F$, G be a group, and y be an element of the carrier of G. If $i \in I$ and $G = F(i)$ and $f = x$ and $g = x^{-1}$ and $f(i) = y$, then $g(i) = y^{-1}$.

Let I be a set and let F be an associative group-like family of semigroups indexed by I. The functor $\operatorname{sum} F$ yielding a strict subgroup of $\prod F$ is defined by the condition (Def. 9).

(Def. 9) Let x be a set. Then $x \in$ the carrier of sum F if and only if there exists an element g of $\prod$ (the support of F) and there exists a finite subset J of I and there exists a many sorted set f indexed by J such that $g = 1_{\prod F}$ and $x = g{+\cdot}f$ and for every set j such that $j \in J$ there exists a group-like non empty semigroup G such that $G = F(j)$ and $f(j) \in$ the carrier of G and $f(j) \neq 1_G$.

Let I be a set, let F be an associative group-like family of semigroups indexed by I, and let f, g be elements of the carrier of sum F. One can check that (the multiplication of sum $F)(f, g)$ is function-like and relation-like.

The following proposition is true

(12) For every finite set I and for every associative group-like family F of semigroups indexed by I holds $\prod F = \mathrm{sum}\, F$.

3. The Product of One, Two and Three Groups

One can prove the following proposition

(13) For every non empty semigroup G_1 holds $\langle G_1 \rangle$ is a family of semigroups indexed by $\{1\}$.

Let G_1 be a non empty semigroup. Then $\langle G_1 \rangle$ is a family of semigroups indexed by $\{1\}$.

We now state the proposition

(14) For every group-like non empty semigroup G_1 holds $\langle G_1 \rangle$ is a group-like family of semigroups indexed by $\{1\}$.

Let G_1 be a group-like non empty semigroup. Then $\langle G_1 \rangle$ is a group-like family of semigroups indexed by $\{1\}$.

Next we state the proposition

(15) For every associative non empty semigroup G_1 holds $\langle G_1 \rangle$ is an associative family of semigroups indexed by $\{1\}$.

Let G_1 be an associative non empty semigroup. Then $\langle G_1 \rangle$ is an associative family of semigroups indexed by $\{1\}$.

The following proposition is true

(16) For every commutative non empty semigroup G_1 holds $\langle G_1 \rangle$ is a commutative family of semigroups indexed by $\{1\}$.

Let G_1 be a commutative non empty semigroup. Then $\langle G_1 \rangle$ is a commutative family of semigroups indexed by $\{1\}$.

We now state the proposition

(17) For every group G_1 holds $\langle G_1 \rangle$ is a group-like associative family of semigroups indexed by $\{1\}$.

Let G_1 be a group. Then $\langle G_1 \rangle$ is a group-like associative family of semigroups indexed by $\{1\}$.

Next we state the proposition

(18) Let G_1 be a commutative group. Then $\langle G_1 \rangle$ is a commutative group-like associative family of semigroups indexed by $\{1\}$.

Let G_1 be a commutative group. Then $\langle G_1 \rangle$ is a group-like associative commutative family of semigroups indexed by $\{1\}$.

Let G_1 be a non empty semigroup. Note that every element of $\prod$ the support of $\langle G_1 \rangle$ is finite sequence-like.

Let G_1 be a non empty semigroup. Note that every element of the carrier of $\prod \langle G_1 \rangle$ is finite sequence-like.

Let G_1 be a non empty semigroup and let x be an element of the carrier of G_1. Then $\langle x \rangle$ is an element of $\prod \langle G_1 \rangle$.

One can prove the following proposition

(19) For all non empty semigroups G_1, G_2 holds $\langle G_1, G_2 \rangle$ is a family of semigroups indexed by $\{1, 2\}$.

Let G_1, G_2 be non empty semigroups. Then $\langle G_1, G_2 \rangle$ is a family of semigroups indexed by $\{1, 2\}$.

One can prove the following proposition

(20) For all group-like non empty semigroups G_1, G_2 holds $\langle G_1, G_2 \rangle$ is a group-like family of semigroups indexed by $\{1, 2\}$.

Let G_1, G_2 be group-like non empty semigroups. Then $\langle G_1, G_2 \rangle$ is a group-like family of semigroups indexed by $\{1, 2\}$.

Next we state the proposition

(21) For all associative non empty semigroups G_1, G_2 holds $\langle G_1, G_2 \rangle$ is an associative family of semigroups indexed by $\{1, 2\}$.

Let G_1, G_2 be associative non empty semigroups. Then $\langle G_1, G_2 \rangle$ is an associative family of semigroups indexed by $\{1, 2\}$.

One can prove the following proposition

(22) For all commutative non empty semigroups G_1, G_2 holds $\langle G_1, G_2 \rangle$ is a commutative family of semigroups indexed by $\{1, 2\}$.

Let G_1, G_2 be commutative non empty semigroups. Then $\langle G_1, G_2 \rangle$ is a commutative family of semigroups indexed by $\{1, 2\}$.

The following proposition is true

(23) For all groups G_1, G_2 holds $\langle G_1, G_2 \rangle$ is a group-like associative family of semigroups indexed by $\{1, 2\}$.

Let G_1, G_2 be groups. Then $\langle G_1, G_2 \rangle$ is a group-like associative family of semigroups indexed by $\{1, 2\}$.

Next we state the proposition

(24) Let G_1, G_2 be commutative groups. Then $\langle G_1, G_2 \rangle$ is a group-like associative commutative family of semigroups indexed by $\{1, 2\}$.

Let G_1, G_2 be commutative groups. Then $\langle G_1, G_2 \rangle$ is a group-like associative commutative family of semigroups indexed by $\{1, 2\}$.

Let G_1, G_2 be non empty semigroups. Note that every element of $\prod$ the support of $\langle G_1, G_2 \rangle$ is finite sequence-like.

Let G_1, G_2 be non empty semigroups. Note that every element of the carrier of $\prod \langle G_1, G_2 \rangle$ is finite sequence-like.

Let G_1, G_2 be non empty semigroups, let x be an element of the carrier of G_1, and let y be an element of the carrier of G_2. Then $\langle x, y \rangle$ is an element of $\prod \langle G_1, G_2 \rangle$.

One can prove the following proposition

(25) For all non empty semigroups G_1, G_2, G_3 holds $\langle G_1, G_2, G_3 \rangle$ is a family of semigroups indexed by $\{1, 2, 3\}$.

Let G_1, G_2, G_3 be non empty semigroups. Then $\langle G_1, G_2, G_3 \rangle$ is a family of semigroups indexed by $\{1, 2, 3\}$.

Next we state the proposition

(26) For all group-like non empty semigroups G_1, G_2, G_3 holds $\langle G_1, G_2, G_3 \rangle$ is a group-like family of semigroups indexed by $\{1, 2, 3\}$.

Let G_1, G_2, G_3 be group-like non empty semigroups. Then $\langle G_1, G_2, G_3 \rangle$ is a group-like family of semigroups indexed by $\{1, 2, 3\}$.

Next we state the proposition

(27) Let G_1, G_2, G_3 be associative non empty semigroups. Then $\langle G_1, G_2, G_3 \rangle$ is an associative family of semigroups indexed by $\{1, 2, 3\}$.

Let G_1, G_2, G_3 be associative non empty semigroups. Then $\langle G_1, G_2, G_3 \rangle$ is an associative family of semigroups indexed by $\{1, 2, 3\}$.

One can prove the following proposition

(28) Let G_1, G_2, G_3 be commutative non empty semigroups. Then $\langle G_1, G_2, G_3 \rangle$ is a commutative family of semigroups indexed by $\{1, 2, 3\}$.

Let G_1, G_2, G_3 be commutative non empty semigroups. Then $\langle G_1, G_2, G_3 \rangle$ is a commutative family of semigroups indexed by $\{1, 2, 3\}$.

Next we state the proposition

(29) For all groups G_1, G_2, G_3 holds $\langle G_1, G_2, G_3 \rangle$ is a group-like associative family of semigroups indexed by $\{1, 2, 3\}$.

Let G_1, G_2, G_3 be groups. Then $\langle G_1, G_2, G_3 \rangle$ is a group-like associative family of semigroups indexed by $\{1, 2, 3\}$.

One can prove the following proposition

(30) Let G_1, G_2, G_3 be commutative groups. Then $\langle G_1, G_2, G_3 \rangle$ is a group-like associative commutative family of semigroups indexed by $\{1, 2, 3\}$.

Let G_1, G_2, G_3 be commutative groups. Then $\langle G_1, G_2, G_3 \rangle$ is a group-like associative commutative family of semigroups indexed by $\{1, 2, 3\}$.

Let G_1, G_2, G_3 be non empty semigroups. Observe that every element of $\prod$ the support of $\langle G_1, G_2, G_3 \rangle$ is finite sequence-like.

Let G_1, G_2, G_3 be non empty semigroups. Note that every element of the carrier of $\prod \langle G_1, G_2, G_3 \rangle$ is finite sequence-like.

Let G_1, G_2, G_3 be non empty semigroups, let x be an element of the carrier of G_1, let y be an element of the carrier of G_2, and let z be an element of the carrier of G_3. Then $\langle x, y, z \rangle$ is an element of $\prod \langle G_1, G_2, G_3 \rangle$.

For simplicity, we adopt the following rules: G_1, G_2, G_3 denote non empty semigroups, x_1, x_2 denote elements of the carrier of G_1, y_1, y_2 denote elements of the carrier of G_2, and z_1, z_2 denote elements of the carrier of G_3.

One can prove the following propositions:

(31) $\langle x_1 \rangle \cdot \langle x_2 \rangle = \langle x_1 \cdot x_2 \rangle$.

(32) $\langle x_1, y_1 \rangle \cdot \langle x_2, y_2 \rangle = \langle x_1 \cdot x_2, y_1 \cdot y_2 \rangle$.

(33) $\langle x_1, y_1, z_1 \rangle \cdot \langle x_2, y_2, z_2 \rangle = \langle x_1 \cdot x_2, y_1 \cdot y_2, z_1 \cdot z_2 \rangle$.

In the sequel G_1, G_2, G_3 denote group-like non empty semigroups.

We now state three propositions:

(34) $1_{\prod \langle G_1 \rangle} = \langle 1_{(G_1)} \rangle$.

(35) $1_{\prod \langle G_1, G_2 \rangle} = \langle 1_{(G_1)}, 1_{(G_2)} \rangle$.

(36) $1_{\prod \langle G_1, G_2, G_3 \rangle} = \langle 1_{(G_1)}, 1_{(G_2)}, 1_{(G_3)} \rangle$.

For simplicity, we adopt the following rules: G_1, G_2, G_3 are groups, x is an element of the carrier of G_1, y is an element of the carrier of G_2, and z is an element of the carrier of G_3.

The following propositions are true:

(37) $(\langle x \rangle$ **qua** element of the carrier of $\prod \langle G_1 \rangle)^{-1} = \langle x^{-1} \rangle$.

(38) $(\langle x, y \rangle$ **qua** element of the carrier of $\prod \langle G_1, G_2 \rangle)^{-1} = \langle x^{-1}, y^{-1} \rangle$.

(39) $(\langle x, y, z \rangle$ **qua** element of the carrier of $\prod \langle G_1, G_2, G_3 \rangle)^{-1} = \langle x^{-1}, y^{-1}, z^{-1} \rangle$.

(40) Let f be a function from the carrier of G_1 into the carrier of $\prod \langle G_1 \rangle$. Suppose that for every element x of the carrier of G_1 holds $f(x) = \langle x \rangle$. Then f is a homomorphism from G_1 to $\prod \langle G_1 \rangle$.

(41) Let f be a homomorphism from G_1 to $\prod \langle G_1 \rangle$. Suppose that for every element x of the carrier of G_1 holds $f(x) = \langle x \rangle$. Then f is an isomorphism.

(42) G_1 and $\prod \langle G_1 \rangle$ are isomorphic.

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