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A Decomposition of a Simple Closed Curves and the Order of Their Points

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Summary. The goal of the article is to introduce an order on a simple closed curve. To do this, we fix two points on the curve and devide it into two arcs. We prove that such a decomposition is unique. Other auxiliary theorems about arcs are proven for preparation of the proof of the above.

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The papers [41], [46], [45], [40], [26], [1], [49], [44], [37], [12], [39], [10], [36], [32], [48], [2], [7], [8], [4], [20], [21], [34], [33], [29], [11], [43], [28], [19], [35], [16], [9], [15], [42], [18], [22], [17], [6], [23], [27], [3], [31], [5], [38], [13], [25], [47], [14], [30], and [24] provide the notation and terminology for this paper.

1. Middle Points of Arcs

For simplicity, we use the following convention: a, b, c, s, r are real numbers, n is a natural number, p, q are points of $\mathcal{E}^2_{\rm T}$, and P is a subset of the carrier of $\mathcal{E}^2_{\rm T}$.

The following propositions are true:

(1) If $a = \frac{a+b}{2}$, then $a = b$.

(2) If $r \leqslant s$, then $r \leqslant \frac{r+s}{2}$ and $\frac{r+s}{2} \leqslant s$.

(3) Let T_1 be a non empty topological space, P be a subset of the carrier of T_1, A be a subset of the carrier of $T_1 \restriction P$, and B be a subset of the carrier of T_1. If B is closed and $A = B \cap P$, then A is closed.

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(4) Let T_1, T_2 be non empty topological spaces, P be a non empty subset of the carrier of T_2, and f be a map from T_1 into $T_2{\restriction}P$. Then

(i) f is a map from T_1 into T_2, and

(ii) for every map f_2 from T_1 into T_2 such that $f_2 = f$ and f is continuous holds f_2 is continuous.

(5) Let r be a real number and P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. If $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{1}} \geqslant r\}$, then P is closed.

(6) Let r be a real number and P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. If $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{1}} \leqslant r\}$, then P is closed.

(7) Let r be a real number and P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. If $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{1}} = r\}$, then P is closed.

(8) Let r be a real number and P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. If $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{2}} \geqslant r\}$, then P is closed.

(9) Let r be a real number and P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. If $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{2}} \leqslant r\}$, then P is closed.

(10) Let r be a real number and P be a subset of the carrier of $\mathcal{E}^2_{\rm T}$. If $P = \{p; p$ ranges over points of $\mathcal{E}^2_{\rm T}$: $p_{\mathbf{2}} = r\}$, then P is closed.

(11) Let P be a non empty subset of the carrier of $\mathcal{E}^n_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. If P is an arc from p_1 to p_2, then P is connected.

(12) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2, then P is closed.

(13) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2. Then there exists a point q of $\mathcal{E}^2_{\rm T}$ such that $q \in P$ and $q_{\mathbf{1}} = \frac{(p_1)_{\mathbf{1}}+(p_2)_{\mathbf{1}}}{2}$.

(14) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$, Q be a subset of the carrier of $\mathcal{E}^2_{\rm T}$, and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and $Q = \{q : q_{\mathbf{1}} = \frac{(p_1)_{\mathbf{1}}+(p_2)_{\mathbf{1}}}{2}\}$. Then P meets Q and $P \cap Q$ is closed.

(15) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$, Q be a subset of the carrier of $\mathcal{E}^2_{\rm T}$, and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is an arc from p_1 to p_2 and $Q = \{q : q_{\mathbf{2}} = \frac{(p_1)_{\mathbf{2}}+(p_2)_{\mathbf{2}}}{2}\}$. Then P meets Q and $P \cap Q$ is closed.

Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Let us assume that P is an arc from p_1 to p_2. The functor xMiddle(P, p_1, p_2) yields a point of $\mathcal{E}^2_{\rm T}$ and is defined as follows:

(Def. 1) For every subset Q of the carrier of $\mathcal{E}^2_{\rm T}$ such that $Q = \{q : q_{\mathbf{1}} = \frac{(p_1)_{\mathbf{1}}+(p_2)_{\mathbf{1}}}{2}\}$ holds xMiddle$(P, p_1, p_2) =$ FPoint(P, p_1, p_2, Q).

Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and let p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Let us assume that P is an arc from p_1 to p_2. The functor yMiddle(P, p_1, p_2) yields a point of $\mathcal{E}^2_{\rm T}$ and is defined by:

(Def. 2) For every subset Q of the carrier of $\mathcal{E}^2_{\rm T}$ such that $Q = \{q : q_{\bf 2} = \frac{(p_1)_{\bf 2}+(p_2)_{\bf 2}}{2}\}$ holds yMiddle$(P, p_1, p_2) =$ FPoint(P, p_1, p_2, Q).

One can prove the following propositions:

(16) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2, then xMiddle$(P, p_1, p_2) \in P$ and yMiddle$(P, p_1, p_2) \in P$.

(17) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2, then $p_1 =$ xMiddle(P, p_1, p_2) iff $(p_1)_{\bf 1} = (p_2)_{\bf 1}$.

(18) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2, then $p_1 =$ yMiddle(P, p_1, p_2) iff $(p_1)_{\bf 2} = (p_2)_{\bf 2}$.

2. Segments of Arcs

The following proposition is true

(19) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2 and LE q_1, q_2, P, p_1, p_2, then LE q_2, q_1, P, p_2, p_1.

Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and let p_1, p_2, q_1 be points of $\mathcal{E}^2_{\rm T}$. The functor LSegment(P, p_1, p_2, q_1) yields a subset of the carrier of $\mathcal{E}^2_{\rm T}$ and is defined by:

(Def. 3) LSegment$(P, p_1, p_2, q_1) = \{q :$ LE q, q_1, P, p_1, $p_2\}$.

Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and let p_1, p_2, q_1 be points of $\mathcal{E}^2_{\rm T}$. The functor RSegment(P, p_1, p_2, q_1) yielding a subset of the carrier of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def. 4) RSegment$(P, p_1, p_2, q_1) = \{q :$ LE q_1, q, P, p_1, $p_2\}$.

Next we state several propositions:

(20) For every non empty subset P of the carrier of $\mathcal{E}^2_{\rm T}$ and for all points p_1, p_2, q_1 of $\mathcal{E}^2_{\rm T}$ holds LSegment$(P, p_1, p_2, q_1) \subseteq P$.

(21) For every non empty subset P of the carrier of $\mathcal{E}^2_{\rm T}$ and for all points p_1, p_2, q_1 of $\mathcal{E}^2_{\rm T}$ holds RSegment$(P, p_1, p_2, q_1) \subseteq P$.

(22) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2, then LSegment$(P, p_1, p_2, p_1) = \{p_1\}$.

(23) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2 and $q \in P$, then LE q, p_2, P, p_1, p_2.

(24) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\rm T}$. If P is an arc from p_1 to p_2 and $q \in P$, then LE p_1, q, P, p_1, p_2.

(25) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2, then $\mathrm{LSegment}(P, p_1, p_2, p_2) = P$.

(26) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2, then $\mathrm{RSegment}(P, p_1, p_2, p_2) = \{p_2\}$.

(27) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2, then $\mathrm{RSegment}(P, p_1, p_2, p_1) = P$.

(28) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $q_1 \in P$, then $\mathrm{RSegment}(P, p_1, p_2, q_1) = \mathrm{LSegment}(P, p_2, p_1, q_1)$.

Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and let p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. The functor $\mathrm{Segment}(P, p_1, p_2, q_1, q_2)$ yielding a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is defined by:

(Def. 5) $\mathrm{Segment}(P, p_1, p_2, q_1, q_2) = \mathrm{RSegment}(P, p_1, p_2, q_1) \cap \mathrm{LSegment}(P, p_1, p_2, q_2)$.

Next we state four propositions:

(29) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Then $\mathrm{Segment}(P, p_1, p_2, q_1, q_2) = \{q : \mathrm{LE}\ q_1, q, P, p_1, p_2 \wedge \mathrm{LE}\ q, q_2, P, p_1, p_2\}$.

(30) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is an arc from p_1 to p_2. Then LE q_1, q_2, P, p_1, p_2 if and only if LE q_2, q_1, P, p_2, p_1.

(31) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $q \in P$, then $\mathrm{LSegment}(P, p_1, p_2, q) = \mathrm{RSegment}(P, p_2, p_1, q)$.

(32) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and $q_1 \in P$ and $q_2 \in P$, then $\mathrm{Segment}(P, p_1, p_2, q_1, q_2) = \mathrm{Segment}(P, p_2, p_1, q_2, q_1)$.

3. Decomposition of a Simple Closed Curve Into Two Arcs

Let s be a real number. The functor $\mathrm{VerticalLine}\, s$ yields a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and is defined as follows:

(Def. 6) $\mathrm{VerticalLine}\, s = \{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{1}} = s\}$.

The functor $\mathrm{HorizontalLine}\, s$ yielding a subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ is defined as follows:

(Def. 7) $\mathrm{HorizontalLine}\, s = \{p : p_{\mathbf{2}} = s\}$.

Next we state several propositions:

(33) For every real number r holds $\mathrm{VerticalLine}\, r$ is closed and $\mathrm{HorizontalLine}\, r$ is closed.

(34) For every real number r and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ VerticalLine r holds $p_{\mathbf{1}} = r$.

(35) For every real number r and for every point p of $\mathcal{E}^2_{\mathrm{T}}$ such that $p \in$ HorizontalLine r holds $p_{\mathbf{2}} = r$.

(36) For every compact non empty subset P of $\mathcal{E}^2_{\mathrm{T}}$ holds W-min $P \in P$ and W-max $P \in P$.

(37) For every compact non empty subset P of $\mathcal{E}^2_{\mathrm{T}}$ holds N-min $P \in P$ and N-max $P \in P$.

(38) For every compact non empty subset P of $\mathcal{E}^2_{\mathrm{T}}$ holds E-min $P \in P$ and E-max $P \in P$.

(39) For every compact non empty subset P of $\mathcal{E}^2_{\mathrm{T}}$ holds S-min $P \in P$ and S-max $P \in P$.

(40) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is a simple closed curve. Then there exist non empty subsets P_1, P_2 of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ such that

(i) P_1 is an arc from W-min P to E-max P,

(ii) P_2 is an arc from E-max P to W-min P,

(iii) $P_1 \cap P_2 = \{\text{W-min}\, P, \text{E-max}\, P\}$,

(iv) $P_1 \cup P_2 = P$, and

(v) $(\mathrm{FPoint}(P_1, \text{W-min}\, P, \text{E-max}\, P, \mathrm{VerticalLine}\, \frac{\text{W-bound}\, P + \text{E-bound}\, P}{2}))_{\mathbf{2}} > (\mathrm{LPoint}(P_2, \text{E-max}\, P, \text{W-min}\, P, \mathrm{VerticalLine}\, \frac{\text{W-bound}\, P + \text{E-bound}\, P}{2}))_{\mathbf{2}}$.

4. Uniqueness of Decomposition of a Simple Closed Curve

One can prove the following propositions:

(41) For every subset P of the carrier of $\mathbb{I}$ such that $P =$ (the carrier of $\mathbb{I}) \setminus \{0, 1\}$ holds P is open.

(42) For all subsets B_1, B_2 of $\mathbb{R}$ such that B_2 is lower bounded and $B_1 \subseteq B_2$ holds B_1 is lower bounded.

(43) For all subsets B_1, B_2 of $\mathbb{R}$ such that B_2 is upper bounded and $B_1 \subseteq B_2$ holds B_1 is upper bounded.

(44) For all r, s holds $]r, s[\cap \{r, s\} = \emptyset$.

(45) For all a, b, c holds $c \in]a, b[$ iff $a < c$ and $c < b$.

(46) For every subset P of the carrier of $\mathbb{R}^{\mathbf{1}}$ and for all r, s such that $P =]r, s[$ holds P is open.

(47) Let S be a non empty topological space, P_1, P_2 be subsets of the carrier of S, and P_1' be a subset of the carrier of $S \restriction P_2$. If $P_1 = P_1'$ and $P_1 \neq \emptyset$ and $P_1 \subseteq P_2$, then $S \restriction P_1 = S \restriction P_2 \restriction P_1'$.

(48) For every subset P_7 of the carrier of $\mathbb{I}$ such that $P_7 =$ (the carrier of $\mathbb{I}) \setminus \{0, 1\}$ holds $P_7 \neq \emptyset$ and P_7 is connected.

(49) Let P be a non empty subset of the carrier of $\mathcal{E}^n_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. If P is an arc from p_1 to p_2, then $p_1 \neq p_2$.

(50) Let P be a non empty subset of the carrier of $\mathcal{E}^n_{\rm T}$, Q be a subset of the carrier of $(\mathcal{E}^n_{\rm T}) \restriction P$, and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. If P is an arc from p_1 to p_2 and $Q = P \setminus \{p_1, p_2\}$, then Q is open.

(51) For all points p, q of $\mathcal{E}^n_{\rm T}$ and for every non empty subset P of $\mathcal{E}^n_{\rm T}$ such that P is an arc from p to q holds P is compact.

(52) Let P be a subset of the carrier of $\mathcal{E}^n_{\rm T}$, P_1, P_2 be non empty subsets of the carrier of $\mathcal{E}^n_{\rm T}$, Q be a subset of the carrier of $(\mathcal{E}^n_{\rm T}) \restriction P$, and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. Suppose $p_1 \in P$ and $p_2 \in P$ and P_1 is an arc from p_1 to p_2 and P_2 is an arc from p_1 to p_2 and $P_1 \cup P_2 = P$ and $P_1 \cap P_2 = \{p_1, p_2\}$ and $Q = P_1 \setminus \{p_1, p_2\}$. Then Q is open.

(53) Let P be a non empty subset of the carrier of $\mathcal{E}^n_{\rm T}$, Q be a subset of the carrier of $(\mathcal{E}^n_{\rm T}) \restriction P$, and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. If P is an arc from p_1 to p_2 and $Q = P \setminus \{p_1, p_2\}$, then Q is connected.

(54) Let G_1 be a non empty topological space, P_1, P be non empty subsets of the carrier of G_1, Q' be a subset of the carrier of $G_1 \restriction P_1$, and Q be a non empty subset of the carrier of $G_1 \restriction P$. If $P_1 \subseteq P$ and $Q = Q'$ and Q' is connected, then Q is connected.

(55) Let P be a non empty subset of the carrier of $\mathcal{E}^n_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. Suppose P is an arc from p_1 to p_2. Then there exists a point p_3 of $\mathcal{E}^n_{\rm T}$ such that $p_3 \in P$ and $p_3 \neq p_1$ and $p_3 \neq p_2$.

(56) Let P be a non empty subset of the carrier of $\mathcal{E}^n_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. If P is an arc from p_1 to p_2, then $P \setminus \{p_1, p_2\} \neq \emptyset$.

(57) Let P_1 be a non empty subset of the carrier of $\mathcal{E}^n_{\rm T}$, P be a subset of the carrier of $\mathcal{E}^n_{\rm T}$, Q be a subset of the carrier of $(\mathcal{E}^n_{\rm T}) \restriction P$, and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. If P_1 is an arc from p_1 to p_2 and $P_1 \subseteq P$ and $Q = P_1 \setminus \{p_1, p_2\}$, then Q is connected.

(58) Let T, S, V be non empty topological spaces, P_1 be a non empty subset of the carrier of S, P_2 be a subset of the carrier of S, f be a map from T into $S \restriction P_1$, and g be a map from $S \restriction P_2$ into V. Suppose $P_1 \subseteq P_2$ and f is continuous and g is continuous. Then there exists a map h from T into V such that $h = g \cdot f$ and h is continuous.

(59) Let P_1, P_2 be non empty subsets of the carrier of $\mathcal{E}^n_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$. If P_1 is an arc from p_1 to p_2 and P_2 is an arc from p_1 to p_2 and $P_1 \subseteq P_2$, then $P_1 = P_2$.

(60) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$, Q be a subset of the carrier of $(\mathcal{E}^2_{\rm T}) \restriction P$, and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose P is a simple closed

curve and $p_1 \in P$ and $p_2 \in P$ and $p_1 \neq p_2$ and $Q = P \setminus \{p_1, p_2\}$. Then Q is not connected.

(61) Let P, P_1, P_2, P'_1, P'_2 be non empty subsets of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose that

(i) P is a simple closed curve,
(ii) P_1 is an arc from p_1 to p_2,
(iii) P_2 is an arc from p_1 to p_2,
(iv) $P_1 \cup P_2 = P$,
(v) $P_1 \cap P_2 = \{p_1, p_2\}$,
(vi) P'_1 is an arc from p_1 to p_2,
(vii) P'_2 is an arc from p_1 to p_2,
(viii) $P'_1 \cup P'_2 = P$, and
(ix) $P'_1 \cap P'_2 = \{p_1, p_2\}$.

Then $P_1 = P'_1$ and $P_2 = P'_2$ or $P_1 = P'_2$ and $P_2 = P'_1$.

5. Lower Arcs and Upper Arcs

One can prove the following propositions:

(62) Let P_1 be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. If P_1 is an arc from p_1 to p_2, then P_1 is closed.

(63) Let G_1, G_2 be non empty topological spaces, P be a non empty subset of the carrier of G_2, f be a map from G_1 into $G_2 \restriction P$, and f_1 be a map from G_1 into G_2. If $f = f_1$ and f is continuous, then f_1 is continuous.

(64) Let P_1 be a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and p_1, p_2 be points of $\mathcal{E}^2_{\rm T}$. Suppose $(p_1)_{\mathbf{1}} \leqslant (p_2)_{\mathbf{1}}$ and P_1 is an arc from p_1 to p_2. Then $P_1 \cap \mathrm{VerticalLine}\,\frac{(p_1)_{\mathbf{1}}+(p_2)_{\mathbf{1}}}{2} \neq \emptyset$ and $P_1 \cap \mathrm{VerticalLine}\,\frac{(p_1)_{\mathbf{1}}+(p_2)_{\mathbf{1}}}{2}$ is closed.

Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Let us assume that P is a simple closed curve. The functor $\mathrm{UpperArc}\,P$ yields a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ and is defined by the conditions (Def. 8).

(Def. 8)(i) $\mathrm{UpperArc}\,P$ is an arc from $\mathrm{W\text{-}min}\,P$ to $\mathrm{E\text{-}max}\,P$, and

(ii) there exists a non empty subset P_2 of the carrier of $\mathcal{E}^2_{\rm T}$ such that P_2 is an arc from $\mathrm{E\text{-}max}\,P$ to $\mathrm{W\text{-}min}\,P$ and $\mathrm{UpperArc}\,P \cap P_2 = \{\mathrm{W\text{-}min}\,P, \mathrm{E\text{-}max}\,P\}$ and $\mathrm{UpperArc}\,P \cup P_2 = P$ and $(\mathrm{FPoint}(\mathrm{UpperArc}\,P, \mathrm{W\text{-}min}\,P, \mathrm{E\text{-}max}\,P, \mathrm{VerticalLine}\,\frac{\mathrm{W\text{-}bound}\,P+\mathrm{E\text{-}bound}\,P}{2}))_{\mathbf{2}} > (\mathrm{LPoint}(P_2, \mathrm{E\text{-}max}\,P, \mathrm{W\text{-}min}\,P, \mathrm{VerticalLine}\,\frac{\mathrm{W\text{-}bound}\,P+\mathrm{E\text{-}bound}\,P}{2}))_{\mathbf{2}}$.

Let P be a compact non empty subset of $\mathcal{E}^2_{\rm T}$. Let us assume that P is a simple closed curve. The functor $\mathrm{LowerArc}\,P$ yielding a non empty subset of the carrier of $\mathcal{E}^2_{\rm T}$ is defined as follows:

(Def. 9) LowerArc P is an arc from E-max P to W-min P and UpperArc $P \cap$ LowerArc $P = \{$W-min P, E-max $P\}$ and UpperArc $P \cup$ LowerArc $P = P$ and (FPoint(UpperArc P, W-min P, E-max P, VerticalLine $\frac{\text{W-bound } P + \text{E-bound } P}{2}$))$_{\mathbf{2}}$ > (LPoint(LowerArc P, E-max P, W-min P, VerticalLine $\frac{\text{W-bound } P + \text{E-bound } P}{2}$))$_{\mathbf{2}}$.

The following propositions are true:

(65) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. Suppose P is a simple closed curve. Then

(i) UpperArc P is an arc from W-min P to E-max P,
(ii) UpperArc P is an arc from E-max P to W-min P,
(iii) LowerArc P is an arc from E-max P to W-min P,
(iv) LowerArc P is an arc from W-min P to E-max P,
(v) UpperArc $P \cap$ LowerArc $P = \{$W-min P, E-max $P\}$,
(vi) UpperArc $P \cup$ LowerArc $P = P$, and
(vii) (FPoint(UpperArc P, W-min P, E-max P, VerticalLine $\frac{\text{W-bound } P + \text{E-bound } P}{2}$))$_{\mathbf{2}}$ > (LPoint(LowerArc P, E-max P, W-min P, VerticalLine $\frac{\text{W-bound } P + \text{E-bound } P}{2}$))$_{\mathbf{2}}$.

(66) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$. If P is a simple closed curve, then LowerArc $P = (P \setminus$ UpperArc $P) \cup \{$W-min P, E-max $P\}$ and UpperArc $P = (P \setminus$ LowerArc $P) \cup \{$W-min P, E-max $P\}$.

(67) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and P_1 be a subset of the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction P$. If P is a simple closed curve and UpperArc $P \cap P_1 = \{$W-min P, E-max $P\}$ and UpperArc $P \cup P_1 = P$, then $P_1 =$ LowerArc P.

(68) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and P_1 be a subset of the carrier of $(\mathcal{E}^2_{\mathrm{T}})\restriction P$. If P is a simple closed curve and $P_1 \cap$ LowerArc $P = \{$W-min P, E-max $P\}$ and $P_1 \cup$ LowerArc $P = P$, then $P_1 =$ UpperArc P.

6. An Order of Points in a Simple Closed Curve

One can prove the following propositions:

(69) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and LE q, p_1, P, p_1, p_2, then $q = p_1$.

(70) Let P be a non empty subset of the carrier of $\mathcal{E}^2_{\mathrm{T}}$ and p_1, p_2, q be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is an arc from p_1 to p_2 and LE p_2, q, P, p_1, p_2, then $q = p_2$.

Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and let q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. The predicate LE$(q_1, q_2, P)$ is defined by the conditions (Def. 10).

(Def. 10)(i) $q_1 \in$ UpperArc P and $q_2 \in$ LowerArc P and $q_2 \neq$ W-min P, or
(ii) $q_1 \in$ UpperArc P and $q_2 \in$ UpperArc P and LE q_1, q_2, UpperArc P, W-min P, E-max P, or

(iii) $q_1 \in \mathrm{LowerArc}\, P$ and $q_2 \in \mathrm{LowerArc}\, P$ and $q_2 \neq$ W-min P and LE q_1, q_2, LowerArc P, E-max P, W-min P.

Next we state three propositions:

(71) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and q be a point of $\mathcal{E}^2_{\mathrm{T}}$. If P is a simple closed curve and $q \in P$, then $\mathrm{LE}(q, q, P)$.

(72) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and q_1, q_2 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is a simple closed curve and $\mathrm{LE}(q_1, q_2, P)$ and $\mathrm{LE}(q_2, q_1, P)$, then $q_1 = q_2$.

(73) Let P be a compact non empty subset of $\mathcal{E}^2_{\mathrm{T}}$ and q_1, q_2, q_3 be points of $\mathcal{E}^2_{\mathrm{T}}$. If P is a simple closed curve and $\mathrm{LE}(q_1, q_2, P)$ and $\mathrm{LE}(q_2, q_3, P)$, then $\mathrm{LE}(q_1, q_3, P)$.

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