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# Algebraic Lattices

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The articles [18], [20], [16], [10], [21], [7], [19], [13], [8], [1], [17], [2], [3], [12], [22], [4], [9], [6], [11], [14], [5], and [15] provide the notation and terminology for this paper.

## 1. The Subset of All Compact Elements

Let $L$ be a non empty reflexive relational structure. The functor CompactSublatt($L$) yields a strict full relational substructure of $L$ and is defined as follows:

(Def. 1) For every element $x$ of $L$ holds $x \in$ the carrier of CompactSublatt($L$) iff $x$ is compact.

Let $L$ be a lower-bounded non empty reflexive antisymmetric relational structure. Observe that CompactSublatt($L$) is non empty.

Next we state three propositions:

(1) Let $L$ be a complete lattice and $x$, $y$, $k$ be elements of $L$. If $x \leqslant k$ and $k \leqslant y$ and $k \in$ the carrier of CompactSublatt($L$), then $x \ll y$.

(2) Let $L$ be a complete lattice and $x$ be an element of $L$. Then $\uparrow x$ is an open filter of $L$ if and only if $x$ is compact.

(3) For every lower-bounded non empty poset $L$ with l.u.b.'s holds CompactSublatt($L$) is join-inheriting and $\perp_L \in$ the carrier of CompactSublatt($L$).

Let $L$ be a non empty reflexive relational structure and let $x$ be an element of $L$. The functor compactbelow($x$) yielding a subset of $L$ is defined by:

(Def. 2) compactbelow($x$) $= \{y, y$ ranges over elements of $L$: $x \geqslant y \ \wedge \ y$ is compact$\}$.




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We now state three propositions:

(4) Let $L$ be a non empty reflexive relational structure and $x$, $y$ be elements of $L$. Then $y \in \mathrm{compactbelow}(x)$ if and only if the following conditions are satisfied:

(i) $x \geqslant y$, and

(ii) $y$ is compact.

(5) For every non empty reflexive relational structure $L$ and for every element $x$ of $L$ holds $\mathrm{compactbelow}(x) = {\downarrow}x \cap$ the carrier of CompactSublatt($L$).

(6) For every non empty reflexive transitive relational structure $L$ and for every element $x$ of $L$ holds $\mathrm{compactbelow}(x) \subseteq {\downarrow}x$.

Let $L$ be a non empty lower-bounded reflexive antisymmetric relational structure and let $x$ be an element of $L$. Note that compactbelow($x$) is non empty.

## 2. Algebraic Lattices

Let $L$ be a non empty reflexive relational structure. We say that $L$ satisfies axiom K if and only if:

(Def. 3) For every element $x$ of $L$ holds $x = \sup \mathrm{compactbelow}(x)$.

Let $L$ be a non empty reflexive relational structure. We say that $L$ is algebraic if and only if:

(Def. 4) For every element $x$ of $L$ holds compactbelow($x$) is non empty and directed and $L$ is up-complete and satisfies axiom K.

We now state the proposition

(7) Let $L$ be a lattice. Then $L$ is algebraic if and only if the following conditions are satisfied:

(i) $L$ is continuous, and

(ii) for all elements $x$, $y$ of $L$ such that $x \ll y$ there exists an element $k$ of $L$ such that $k \in$ the carrier of CompactSublatt($L$) and $x \leqslant k$ and $k \leqslant y$.

Let us observe that every lattice which is algebraic is also continuous.

Let us note that every non empty reflexive relational structure which is algebraic is also up-complete and satisfies axiom K.

Let $L$ be a non empty poset with l.u.b.'s. One can check that CompactSublatt($L$) is join-inheriting.

Let $L$ be a lattice. We say that $L$ is arithmetic if and only if:

(Def. 5) $L$ is algebraic and CompactSublatt($L$) is meet-inheriting.

## 3. Arithmetic Lattices

Let us note that every lattice which is arithmetic is also algebraic.

Let us note that every lattice which is trivial is also arithmetic.

Let us note that there exists a lattice which is trivial and strict.

We now state a number of propositions:

(8) Let $L_1$, $L_2$ be non empty reflexive antisymmetric relational structures. Suppose the relational structure of $L_1$ = the relational structure of $L_2$ and $L_1$ is up-complete. Let $x_1$, $y_1$ be elements of $L_1$ and $x_2$, $y_2$ be elements of $L_2$. If $x_1 = x_2$ and $y_1 = y_2$ and $x_1 \ll y_1$, then $x_2 \ll y_2$.

(9) Let $L_1$, $L_2$ be non empty reflexive antisymmetric relational structures. Suppose the relational structure of $L_1$ = the relational structure of $L_2$ and $L_1$ is up-complete. Let $x$ be an element of $L_1$ and $y$ be an element of $L_2$. If $x = y$ and $x$ is compact, then $y$ is compact.

(10) Let $L_1$, $L_2$ be up-complete non empty reflexive antisymmetric relational structures. Suppose the relational structure of $L_1$ = the relational structure of $L_2$. Let $x$ be an element of $L_1$ and $y$ be an element of $L_2$. If $x = y$, then $\mathrm{compactbelow}(x) = \mathrm{compactbelow}(y)$.

(11) Let $L_1$, $L_2$ be relational structures. Suppose the relational structure of $L_1$ = the relational structure of $L_2$ and $L_1$ is non empty. Then $L_2$ is non empty.

(12) Let $L_1$, $L_2$ be non empty relational structures. Suppose the relational structure of $L_1$ = the relational structure of $L_2$ and $L_1$ is reflexive. Then $L_2$ is reflexive.

(13) Let $L_1$, $L_2$ be relational structures. Suppose the relational structure of $L_1$ = the relational structure of $L_2$ and $L_1$ is transitive. Then $L_2$ is transitive.

(14) Let $L_1$, $L_2$ be relational structures. Suppose the relational structure of $L_1$ = the relational structure of $L_2$ and $L_1$ is antisymmetric. Then $L_2$ is antisymmetric.

(15) Let $L_1$, $L_2$ be non empty reflexive relational structures. Suppose the relational structure of $L_1$ = the relational structure of $L_2$ and $L_1$ is up-complete. Then $L_2$ is up-complete.

(16) For all up-complete non empty reflexive antisymmetric relational structures $L_1$, $L_2$ such that the relational structure of $L_1$ = the relational structure of $L_2$ and $L_1$ satisfies axiom K and for every element $x$ of $L_1$ holds $\mathrm{compactbelow}(x)$ is non empty and directed holds $L_2$ satisfies axiom K.

(17) Let $L_1$, $L_2$ be non empty reflexive antisymmetric relational structures. Suppose the relational structure of $L_1$ = the relational structure of $L_2$ and $L_1$ is algebraic. Then $L_2$ is algebraic.

(18) Let $L_1$, $L_2$ be lattices. Suppose the relational structure of $L_1$ = the relational structure of $L_2$ and $L_1$ is arithmetic. Then $L_2$ is arithmetic.

Let $L$ be a non empty relational structure. Observe that the relational structure of $L$ is non empty.

Let $L$ be a non empty reflexive relational structure. One can check that the relational structure of $L$ is reflexive.

Let $L$ be a transitive relational structure. Note that the relational structure of $L$ is transitive.

Let $L$ be an antisymmetric relational structure. Observe that the relational structure of $L$ is antisymmetric.

Let $L$ be a relational structure with g.l.b.'s. Note that the relational structure of $L$ has g.l.b.'s.

Let $L$ be a relational structure with l.u.b.'s. One can check that the relational structure of $L$ has l.u.b.'s.

Let $L$ be an up-complete non empty reflexive relational structure. One can check that the relational structure of $L$ is up-complete.

Let $L$ be an algebraic non empty reflexive antisymmetric relational structure. Note that the relational structure of $L$ is algebraic.

Let $L$ be an arithmetic lattice. One can verify that the relational structure of $L$ is arithmetic.

Next we state several propositions:

(19) Let $L$ be a non empty transitive relational structure, $S$ be a non empty full relational substructure of $L$, and $X$ be a subset of $S$. Suppose sup $X$ exists in $L$ and $\bigsqcup_L X$ is an element of $S$. Then sup $X$ exists in $S$ and $\sup X = \bigsqcup_L X$.

(20) Let $L$ be a non empty transitive relational structure, $S$ be a non empty full relational substructure of $L$, and $X$ be a subset of $S$. Suppose inf $X$ exists in $L$ and $\bigsqcap_L X$ is an element of $S$. Then inf $X$ exists in $S$ and $\inf X = \bigsqcap_L X$.

(21) For every algebraic lattice $L$ holds $L$ is arithmetic iff CompactSublatt($L$) is a lattice.

(22) For every algebraic lower-bounded lattice $L$ holds $L$ is arithmetic iff $\ll_L$ is multiplicative.

(23) Let $L$ be an arithmetic lower-bounded lattice and $p$ be an element of $L$. If $p$ is pseudoprime, then $p$ is prime.

(24) Let $L$ be an algebraic distributive lower-bounded lattice. Suppose that for every element $p$ of $L$ such that $p$ is pseudoprime holds $p$ is prime. Then $L$ is arithmetic.

Let $L$ be an algebraic lattice and let $c$ be a closure map from $L$ into $L$. Note that there exists a subset of $\operatorname{Im} c$ which is non empty and directed.

We now state three propositions:

(25) Let $L$ be an algebraic lattice and $c$ be a closure map from $L$ into $L$. If $c$ is directed-sups-preserving, then $c^\circ(\Omega_{\mathrm{CompactSublatt}(L)}) \subseteq \Omega_{\mathrm{CompactSublatt}(\operatorname{Im} c)}$.

(26) Let $L$ be an algebraic lower-bounded lattice and $c$ be a closure map from $L$ into $L$. If $c$ is directed-sups-preserving, then $\operatorname{Im} c$ is an algebraic lattice.

(27) Let $L$ be an algebraic lower-bounded lattice and $c$ be a closure map from $L$ into $L$. If $c$ is directed-sups-preserving, then $c^{\circ}(\Omega_{\mathrm{CompactSublatt}(L)}) = \Omega_{\mathrm{CompactSublatt}(\operatorname{Im} c)}$.

## 4. Boolean Posets as Algebraic Lattices

Next we state several propositions:

(28) For all sets $X$, $x$ holds $x$ is an element of $2^X_{\subseteq}$ iff $x \subseteq X$.

(29) Let $X$ be a set and $x$, $y$ be elements of $2^X_{\subseteq}$. Then $x \ll y$ if and only if for every family $Y$ of subsets of $X$ such that $y \subseteq \bigcup Y$ there exists a finite subset $Z$ of $Y$ such that $x \subseteq \bigcup Z$.

(30) For every set $X$ and for every element $x$ of $2^X_{\subseteq}$ holds $x$ is finite iff $x$ is compact.

(31) For every set $X$ and for every element $x$ of $2^X_{\subseteq}$ holds $\mathrm{compactbelow}(x) = \{y : y$ ranges over finite subsets of $x\}$.

(32) For every set $X$ and for every subset $F$ of $X$ holds $F \in$ the carrier of $\mathrm{CompactSublatt}(2^X_{\subseteq})$ iff $F$ is finite.

Let $X$ be a set and let $x$ be an element of $2^X_{\subseteq}$. Observe that $\mathrm{compactbelow}(x)$ is lower and directed.

The following proposition is true

(33) For every set $X$ holds $2^X_{\subseteq}$ is algebraic.

Let $X$ be a set. Observe that $2^X_{\subseteq}$ is algebraic.

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