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Lattice of Congruences in Many Sorted Algebra

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The articles [19], [21], [10], [22], [24], [7], [8], [23], [16], [5], [18], [17], [4], [13], [14], [25], [11], [2], [15], [3], [6], [20], [9], [12], and [1] provide the terminology and notation for this paper.

1. More on Equivalence Relations

For simplicity we adopt the following convention: I, X denote sets, M denotes a many sorted set indexed by I, R_1 denotes a binary relation on X, and E_1, E_2, E_3 denote equivalence relations of X.

We now state the proposition

(1) $(E_1 \sqcup E_2) \sqcup E_3 = E_1 \sqcup (E_2 \sqcup E_3)$.

Let X be a set and let R be a binary relation on X. The functor $\mathrm{EqCl}(R)$ yielding an equivalence relation of X is defined as follows:

(Def. 1) $R \subseteq \mathrm{EqCl}(R)$ and for every equivalence relation E_2 of X such that $R \subseteq E_2$ holds $\mathrm{EqCl}(R) \subseteq E_2$.

One can prove the following propositions:

(2) $E_1 \sqcup E_2 = \mathrm{EqCl}(E_1 \cup E_2)$.

(3) $\mathrm{EqCl}(E_1) = E_1$.

(4) $\nabla_X \cup R_1 = \nabla_X$.

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2. Lattice of Equivalence Relations

Let X be a set. The functor $\mathrm{EqRelLatt}(X)$ yields a strict lattice and is defined by the conditions (Def. 2).

(Def. 2) (i) The carrier of $\mathrm{EqRelLatt}(X) = \{x : x$ ranges over relations between X and X, x is an equivalence relation of $X\}$, and

(ii) for all equivalence relations x, y of X holds (the meet operation of $\mathrm{EqRelLatt}(X))(x, y) = x \cap y$ and (the join operation of $\mathrm{EqRelLatt}(X))(x, y) = x \sqcup y$.

3. Many Sorted Equivalence Relations

Let us consider I, M. Note that there exists a many sorted relation of M which is equivalence.

Let us consider I, M. An equivalence relation of M is an equivalence many sorted relation of M.

We adopt the following convention: I will denote a non empty set, M will denote a many sorted set indexed by I, and E_4, E_1, E_2, E_3 will denote equivalence relations of M.

Let I be a non empty set, let M be a many sorted set indexed by I, and let R be a many sorted relation of M. The functor $\mathrm{EqCl}(R)$ yields an equivalence relation of M and is defined as follows:

(Def. 3) For every element i of I holds $(\mathrm{EqCl}(R))(i) = \mathrm{EqCl}(R(i))$.

The following proposition is true

(5) $\mathrm{EqCl}(E_4) = E_4$.

4. Lattice of Many Sorted Equivalence Relations

Let I be a non empty set, let M be a many sorted set indexed by I, and let E_1, E_2 be equivalence relations of M. The functor $E_1 \sqcup E_2$ yielding an equivalence relation of M is defined as follows:

(Def. 4) There exists a many sorted relation E_3 of M such that $E_3 = E_1 \cup E_2$ and $E_1 \sqcup E_2 = \mathrm{EqCl}(E_3)$.

Let us observe that the functor introduced above is commutative.

Next we state several propositions:

(6) $E_1 \cup E_2 \subseteq E_1 \sqcup E_2$.

(7) For every equivalence relation E_4 of M such that $E_1 \cup E_2 \subseteq E_4$ holds $E_1 \sqcup E_2 \subseteq E_4$.

(8) If $E_1 \cup E_2 \subseteq E_3$ and for every equivalence relation E_4 of M such that $E_1 \cup E_2 \subseteq E_4$ holds $E_3 \subseteq E_4$, then $E_3 = E_1 \sqcup E_2$.

(9) $E_4 \sqcup E_4 = E_4$.

(10) $(E_1 \sqcup E_2) \sqcup E_3 = E_1 \sqcup (E_2 \sqcup E_3)$.

(11) $E_1 \cap (E_1 \sqcup E_2) = E_1$.

(12) For every equivalence relation E_4 of M such that $E_4 = E_1 \cap E_2$ holds $E_1 \sqcup E_4 = E_1$.

(13) For all equivalence relations E_1, E_2 of M holds $E_1 \cap E_2$ is an equivalence relation of M.

Let I be a non empty set and let M be a many sorted set indexed by I. The functor EqRelLatt(M) yielding a strict lattice is defined by the conditions (Def. 5).

(Def. 5) (i) For arbitrary x holds $x \in$ the carrier of EqRelLatt(M) iff x is an equivalence relation of M, and

(ii) for all equivalence relations x, y of M holds (the meet operation of EqRelLatt(M))$(x, y) = x \cap y$ and (the join operation of EqRelLatt(M))$(x, y) = x \sqcup y$.

5. Lattice of Congruences in Many Sorted Algebra

Let S be a non empty many sorted signature and let A be an algebra over S Note that every many sorted relation of A which is equivalence is also equivalence.

In the sequel S will denote a non void non empty many sorted signature and A will denote a non-empty algebra over S.

Next we state several propositions:

(14) Let o be an operation symbol of S, and let C_1, C_2 be congruences of A, and let x_1, y_1 be arbitrary, and let a_1, b_1 be finite sequences. Suppose $\langle x_1, y_1 \rangle \in C_1(\pi_{\mathrm{len}\, a_1+1} \mathrm{Arity}(o)) \cup C_2(\pi_{\mathrm{len}\, a_1+1} \mathrm{Arity}(o))$. Let x, y be elements of Args(o, A). Suppose $x = a_1 \frown \langle x_1 \rangle \frown b_1$ and $y = a_1 \frown \langle y_1 \rangle \frown b_1$. Then $\langle (\mathrm{Den}(o, A))(x), (\mathrm{Den}(o, A))(y) \rangle \in C_1$(the result sort of o)$\cup C_2$(the result sort of o).

(15) Let o be an operation symbol of S, and let C_1, C_2 be congruences of A, and let C be an equivalence many sorted relation of A. Suppose $C = C_1 \sqcup C_2$. Let x_1, y_1 be arbitrary, and let n be a natural number, and let a_1, a_2, b_1 be finite sequences. Suppose $\mathrm{len}\, a_1 = n$ and $\mathrm{len}\, a_1 = \mathrm{len}\, a_2$ and for every natural number k such that $k \in \mathrm{dom}\, a_1$ holds $\langle a_1(k), a_2(k) \rangle \in C(\pi_k \mathrm{Arity}(o))$. Suppose $\langle (\mathrm{Den}(o, A))(a_1 \frown \langle x_1 \rangle \frown b_1), (\mathrm{Den}(o, A))(a_2 \frown \langle x_1 \rangle \frown b_1) \rangle \in C$(the result sort of o) and $\langle x_1, y_1 \rangle \in C(\pi_{n+1} \mathrm{Arity}(o))$. Let x be an element of Args(o, A). If $x = a_1 \frown \langle x_1 \rangle \frown b_1$, then $\langle (\mathrm{Den}(o, A))(x), (\mathrm{Den}(o, A))(a_2 \frown \langle y_1 \rangle \frown b_1) \rangle \in C$(the result sort of o).

(16) Let o be an operation symbol of S, and let C_1, C_2 be congruences of A, and let C be an equivalence many sorted relation of A. Suppose $C = C_1 \sqcup C_2$. Let x, y be elements of $\mathrm{Args}(o, A)$. Suppose that for every natural number n such that $n \in \mathrm{dom}\, x$ holds $\langle x(n), y(n) \rangle \in C(\pi_n \mathrm{Arity}(o))$. Then $\langle (\mathrm{Den}(o, A))(x), (\mathrm{Den}(o, A))(y) \rangle \in C$(the result sort of o).

(17) For all congruences C_1, C_2 of A holds $C_1 \sqcup C_2$ is a congruence of A.

(18) For all congruences C_1, C_2 of A holds $C_1 \cap C_2$ is a congruence of A.

Let us consider S and let A be a non-empty algebra over S. The functor CongrLatt(A) yielding a strict sublattice of EqRelLatt(the sorts of A) is defined by:

(Def. 6) For arbitrary x holds $x \in$ the carrier of CongrLatt(A) iff x is a congruence of A.

We now state four propositions:

(19) $\mathrm{id}_{(\text{the sorts of } A)}$ is a congruence of A.

(20) $[\![$the sorts of A, the sorts of $A]\!]$ is a congruence of A.

(21) $\perp_{\mathrm{CongrLatt}(A)} = \mathrm{id}_{(\text{the sorts of } A)}$.

(22) $\top_{\mathrm{CongrLatt}(A)} = [\![$the sorts of A, the sorts of $A]\!]$.

Let us consider S and let us consider A. One can check that CongrLatt(A) is bounded.

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