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Basic Properties of Functor Structures

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Summary. This article presents some theorems about functor structures. We start with some basic lemmata concerning the composition of functor structures. Then, two theorems about the restriction operator are formulated. Later, we show two theorems stating that the properties 'full' and 'faithful' of functor structures which are equivalent to the 'onto' and 'one-to-one' properties of their morphmaps, respectively. Furthermore, we prove some theorems about the inversion of functor structures.

MML Identifier: FUNCTOR1.

The terminology and notation used here are introduced in the following articles: [17], [16], [6], [18], [4], [5], [3], [15], [14], [9], [8], [11], [12], [2], [13], [10], [7], and [1].

1. Definitions

In this paper X, Y denote sets and Z denotes a non empty set.

Let us mention that there exists a non empty category structure which is transitive and reflexive and has units.

Let A be a non empty reflexive category structure. One can verify that there exists a substructure of A which is non empty and reflexive.

Let C_1, C_2 be non empty reflexive category structures, let F be a feasible functor structure from C_1 to C_2, and let A be a non empty reflexive substructure of C_1. Observe that $F \restriction A$ is feasible.

2. Theorems about sets and functions

We now state four propositions:

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(1) For every set X holds id_X is onto.

(2) Let A be a non empty set, and let B, C be non empty subsets of A and let D be a non empty subset of B. If $C = D$, then $\overset{C}{\hookrightarrow} = (\overset{B}{\hookrightarrow}) \cdot (\overset{D}{\hookrightarrow})$.

(3) For every function f from X into Y such that f is bijective holds f^{-1} is a function from Y into X.

(4) Let f be a function from X into Y and let g be a function from Y into Z. Suppose f is bijective and g is bijective. Then there exists a function h from X into Z such that $h = g \cdot f$ and h is bijective.

3. Theorems about the composition of functor structures

The following propositions are true:

(5) Let A be a non empty reflexive category structure, and let B be a non empty reflexive substructure of A, and let C be a non empty substructure of A, and let D be a non empty substructure of B. If $C = D$, then $\overset{C}{\hookrightarrow} = (\overset{B}{\hookrightarrow}) \cdot (\overset{D}{\hookrightarrow})$.

(6) Let A, B be non empty category structures and let F be a functor structure from A to B. Suppose F is bijective. Then the object map of F is bijective and the morphism map of F is "1-1".

(7) Let C_1 be a non empty graph, and let C_2, C_3 be non empty reflexive graphs, and let F be a feasible functor structure from C_1 to C_2, and let G be a functor structure from C_2 to C_3. If F is one-to-one and G is one-to-one, then $G \cdot F$ is one-to-one.

(8) Let C_1 be a non empty graph, and let C_2, C_3 be non empty reflexive graphs, and let F be a feasible functor structure from C_1 to C_2, and let G be a functor structure from C_2 to C_3 If F is faithful and G is faithful, then $G \cdot F$ is faithful.

(9) Let C_1 be a non empty graph, and let C_2, C_3 be non empty reflexive graphs, and let F be a feasible functor structure from C_1 to C_2, and let G be a functor structure from C_2 to C_3 If F is onto and G is onto, then $G \cdot F$ is onto.

(10) Let C_1 be a non empty graph, and let C_2, C_3 be non empty reflexive graphs, and let F be a feasible functor structure from C_1 to C_2, and let G be a functor structure from C_2 to C_3 If F is full and G is full, then $G \cdot F$ is full.

(11) Let C_1 be a non empty graph, and let C_2, C_3 be non empty reflexive graphs, and let F be a feasible functor structure from C_1 to C_2, and let G be a functor structure from C_2 to C_3 If F is injective and G is injective, then $G \cdot F$ is injective.

(12) Let C_1 be a non empty graph, and let C_2, C_3 be non empty reflexive graphs, and let F be a feasible functor structure from C_1 to C_2, and let G

be a functor structure from C_2 to C_3 If F is surjective and G is surjective, then $G \cdot F$ is surjective.

(13) Let C_1 be a non empty graph, and let C_2, C_3 be non empty reflexive graphs, and let F be a feasible functor structure from C_1 to C_2, and let G be a functor structure from C_2 to C_3 If F is bijective and G is bijective, then $G \cdot F$ is bijective.

4. Theorems about the restriction and inclusion operator

We now state three propositions:

(14) Let A, I be non empty reflexive category structures, and let B be a non empty reflexive substructure of A, and let C be a non empty substructure of A, and let D be a non empty substructure of B. Suppose $C = D$. Let F be a functor structure from A to I. Then $F \restriction C = F \restriction B \restriction D$.

(15) Let C_1, C_2, C_3 be non empty reflexive category structures, and let F be a feasible functor structure from C_1 to C_2, and let G be a functor structure from C_2 to C_3 and let A be a non empty reflexive substructure of C_1. Then $(G \cdot F) \restriction A = G \cdot (F \restriction A)$.

(17)[1] Let A be a non empty category structure and let B be a non empty substructure of A. Then B is full if and only if $\overset{B}{\hookrightarrow}$ is full.

5. Theorems about 'full' and 'faithful' functor structures

Next we state two propositions:

(18) Let C_1, C_2 be non empty category structures and let F be a precovariant functor structure from C_1 to C_2. Then F is full if and only if for all objects o_1, o_2 of C_1 holds $\text{Morph-Map}_F(o_1, o_2)$ is onto.

(19) Let C_1, C_2 be non empty category structures and let F be a precovariant functor structure from C_1 to C_2. Then F is faithful if and only if for all objects o_1, o_2 of C_1 holds $\text{Morph-Map}_F(o_1, o_2)$ is one-to-one.

6. Theorems about the inversion of functor structures

One can prove the following propositions:

(20) For every transitive non empty category structure A with units holds $(\text{id}_A)^{-1} = \text{id}_A$.

[1]The proposition (16) has been removed.

(21) Let A, B be transitive reflexive non empty category structures with units. Suppose A and B are isomorphic. Let F be a strict feasible functor structure from A to B. Suppose F is bijective. Let G be a strict feasible functor structure from B to A. If $G = F^{-1}$, then $F \cdot G = \mathrm{id}_B$.

(22) Let A, B be transitive reflexive non empty category structures with units. Suppose A and B are isomorphic. Let F be a strict feasible functor structure from A to B. If F is bijective, then $F^{-1} \cdot F = \mathrm{id}_A$.

(23) Let A, B be transitive reflexive non empty category structures with units. Suppose A and B are isomorphic. Let F be a strict feasible functor structure from A to B. If F is bijective, then $(F^{-1})^{-1} = F$.

(24) Let A, B, C be transitive reflexive non empty category structures with units, and let G be a strict feasible functor structure from A to B, and let F be a strict feasible functor structure from B to C, and let G_1 be a strict feasible functor structure from B to A, and let F_1 be a strict feasible functor structure from C to B. Suppose F is bijective and G is bijective and F_1 is bijective and G_1 is bijective and $G_1 = G^{-1}$ and $F_1 = F^{-1}$. Then $(F \cdot G)^{-1} = G_1 \cdot F_1$.

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References

[1] Ewa Burakowska. Subalgebras of many sorted algebra. Lattice of subalgebras. *Formalized Mathematics*, 5(**1**):47–54, 1996.

[2] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[7] Artur Korniłowicz. On the group of automorphisms of universal algebra & many sorted algebra. *Formalized Mathematics*, 5(**2**):221–226, 1996.

[8] Małgorzata Korolkiewicz. Homomorphisms of many sorted algebras. *Formalized Mathematics*, 5(**1**):61–65, 1996.

[9] Beata Madras. Product of family of universal algebras. *Formalized Mathematics*, 4(**1**):103–108, 1993.

[10] Yozo Toda. The formalization of simple graphs. *Formalized Mathematics*, 5(**1**):137–144, 1996.

[11] Andrzej Trybulec. Categories without uniqueness of **cod** and **dom**. *Formalized Mathematics*, 5(**2**):259–267, 1996.

[12] Andrzej Trybulec. Examples of category structures. *Formalized Mathematics*, 5(**4**):493–500, 1996.

[13] Andrzej Trybulec. Functors for alternative categories. *Formalized Mathematics*, 5(**4**):595–608, 1996.
[14] Andrzej Trybulec. Many sorted algebras. *Formalized Mathematics*, 5(**1**):37–42, 1996.
[15] Andrzej Trybulec. Many-sorted sets. *Formalized Mathematics*, 4(**1**):15–22, 1993.
[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[17] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.
[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

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