FORMALIZED MATHEMATICS
Volume 5, Number 3, 1996
Warsaw University - Białystok

The Steinitz Theorem and the Dimension of a Vector Space

Mariusz Żynel
Warsaw University
Białystok

Summary. The main purpose of the paper is to define the dimension of an abstract vector space. The dimension of a finite-dimensional vector space is, by the most common definition, the number of vectors in a basis. Obviously, each basis contains the same number of vectors. We prove the Steinitz Theorem together with Exchange Lemma in the second section. The Steinitz Theorem says that each linearly-independent subset of a vector space has cardinality less than any subset that generates the space, moreover it can be extended to a basis. Further we review some of the standard facts involving the dimension of a vector space. Additionally, in the last section, we introduce two notions: the family of subspaces of a fixed dimension and the pencil of subspaces. Both of them can be applied in the algebraic representation of several geometries.

MML Identifier: VECTSP_9.

The terminology and notation used in this paper have been introduced in the following articles: [13], [23], [12], [8], [2], [6], [24], [4], [5], [22], [1], [7], [3], [17], [19], [9], [21], [15], [10], [20], [16], [18], [14], and [11].

1. Preliminaries

For simplicity we follow the rules: G_1 is a field, V is a vector space over G_1, W is a subspace of V, x is arbitrary, and n is a natural number.

Let S be a non empty 1-sorted structure. Observe that there exists a subset of S which is non empty.

One can prove the following proposition

(1) For every finite set X such that $n \leq \overline{\overline{X}}$ there exists a finite subset A of X such that $\overline{\overline{A}} = n$.

ISSN 1426–2630

In the sequel f, g will be functions.

We now state a number of propositions:

(2) For every f such that f is one-to-one holds if $x \in \operatorname{rng} f$, then $\overline{\overline{f^{-1}\{x\}}} = 1$.

(3) For every f such that $x \notin \operatorname{rng} f$ holds $\overline{\overline{f^{-1}\{x\}}} = 0$.

(4) For all f, g such that $\operatorname{rng} f = \operatorname{rng} g$ and f is one-to-one and g is one-to-one holds f and g are fiberwise equipotent.

(5) Let L be a linear combination of V, and let F, G be finite sequences of elements of the carrier of V, and let P be a permutation of $\operatorname{dom} F$. If $G = F \cdot P$, then $\sum(L\, F) = \sum(L\, G)$.

(6) Let L be a linear combination of V and let F be a finite sequence of elements of the carrier of V. If $\operatorname{support} L$ misses $\operatorname{rng} F$, then $\sum(L\, F) = 0_V$.

(7) Let F be a finite sequence of elements of the carrier of V. Suppose F is one-to-one. Let L be a linear combination of V. If $\operatorname{support} L \subseteq \operatorname{rng} F$, then $\sum(L\, F) = \sum L$.

(8) Let L be a linear combination of V and let F be a finite sequence of elements of the carrier of V. Then there exists a linear combination K of V such that $\operatorname{support} K = \operatorname{rng} F \cap \operatorname{support} L$ and $L\, F = K\, F$.

(9) Let L be a linear combination of V, and let A be a subset of V, and let F be a finite sequence of elements of the carrier of V. Suppose $\operatorname{rng} F \subseteq$ the carrier of $\operatorname{Lin}(A)$. Then there exists a linear combination K of A such that $\sum(L\, F) = \sum K$.

(10) Let L be a linear combination of V and let A be a subset of V. Suppose $\operatorname{support} L \subseteq$ the carrier of $\operatorname{Lin}(A)$. Then there exists a linear combination K of A such that $\sum L = \sum K$.

(11) Let L be a linear combination of V. Suppose $\operatorname{support} L \subseteq$ the carrier of W. Let K be a linear combination of W. If $K = L \restriction$ (the carrier of W), then $\operatorname{support} L = \operatorname{support} K$ and $\sum L = \sum K$.

(12) For every linear combination K of W there exists a linear combination L of V such that $\operatorname{support} K = \operatorname{support} L$ and $\sum K = \sum L$.

(13) Let L be a linear combination of V. Suppose $\operatorname{support} L \subseteq$ the carrier of W. Then there exists a linear combination K of W such that $\operatorname{support} K = \operatorname{support} L$ and $\sum K = \sum L$.

(14) For every basis I of V and for every vector v of V holds $v \in \operatorname{Lin}(I)$.

(15) Let A be a subset of W. Suppose A is linearly independent. Then there exists a subset B of V such that B is linearly independent and $B = A$.

(16) Let A be a subset of V. Suppose A is linearly independent and $A \subseteq$ the carrier of W. Then there exists a subset B of W such that B is linearly independent and $B = A$.

(17) For every basis A of W there exists a basis B of V such that $A \subseteq B$.

(18) Let A be a subset of V. Suppose A is linearly independent. Let v be a vector of V. If $v \in A$, then for every subset B of V such that $B = A \setminus \{v\}$

holds $v \notin \mathrm{Lin}(B)$.

(19) Let I be a basis of V and let A be a non empty subset of V. Suppose A misses I. Let B be a subset of V. If $B = I \cup A$, then B is linearly-dependent.

(20) For every subset A of V such that $A \subseteq$ the carrier of W holds $\mathrm{Lin}(A)$ is a subspace of W.

(21) For every subset A of V and for every subset B of W such that $A = B$ holds $\mathrm{Lin}(A) = \mathrm{Lin}(B)$.

2. The Steinitz Theorem

The following two propositions are true:

(22) Let A, B be finite subsets of V and let v be a vector of V. Suppose $v \in \mathrm{Lin}(A \cup B)$ and $v \notin \mathrm{Lin}(B)$. Then there exists a vector w of V such that $w \in A$ and $w \in \mathrm{Lin}(((A \cup B) \setminus \{w\}) \cup \{v\})$.

(23) Let A, B be finite subsets of V. Suppose the vector space structure of $V = \mathrm{Lin}(A)$ and B is linearly independent. Then $\overline{\overline{B}} \leq \overline{\overline{A}}$ and there exists a finite subset C of V such that $C \subseteq A$ and $\overline{\overline{C}} = \overline{\overline{A}} - \overline{\overline{B}}$ and the vector space structure of $V = \mathrm{Lin}(B \cup C)$.

3. Finite-Dimensional Vector Spaces

Let G_1 be a field and let V be a vector space over G_1. Let us observe that V is finite dimensional if and only if:

(Def.1) There exists finite subset of V which is a basis of V.

Next we state several propositions:

(24) If V is finite dimensional, then every basis of V is finite.

(25) If V is finite dimensional, then for every subset A of V such that A is linearly independent holds A is finite.

(26) If V is finite dimensional, then for all bases A, B of V holds $\overline{\overline{A}} = \overline{\overline{B}}$.

(27) $\mathbf{0}_V$ is finite dimensional.

(28) If V is finite dimensional, then W is finite dimensional.

Let G_1 be a field and let V be a vector space over G_1. Observe that there exists a subspace of V which is strict and finite dimensional.

Let G_1 be a field and let V be a finite dimensional vector space over G_1. Note that every subspace of V is finite dimensional.

Let G_1 be a field and let V be a finite dimensional vector space over G_1. One can check that there exists a subspace of V which is strict.

4. The Dimension of a Vector Space

Let G_1 be a field and let V be a vector space over G_1. Let us assume that V is finite dimensional. The functor $\dim(V)$ yields a natural number and is defined by:

(Def.2) For every basis I of V holds $\dim(V) = \overline{\overline{I}}$.

We adopt the following rules: V denotes a finite dimensional vector space over G_1, W, W_1, W_2 denote subspaces of V, and u, v denote vectors of V.

The following propositions are true:

(29) $\dim(W) \leq \dim(V)$.

(30) For every subset A of V such that A is linearly independent holds $\overline{\overline{A}} = \dim(\mathrm{Lin}(A))$.

(31) $\dim(V) = \dim(\Omega_V)$.

(32) $\dim(V) = \dim(W)$ iff $\Omega_V = \Omega_W$.

(33) $\dim(V) = 0$ iff $\Omega_V = \mathbf{0}_V$.

(34) $\dim(V) = 1$ iff there exists v such that $v \neq 0_V$ and $\Omega_V = \mathrm{Lin}(\{v\})$.

(35) $\dim(V) = 2$ iff there exist u, v such that $u \neq v$ and $\{u, v\}$ is linearly independent and $\Omega_V = \mathrm{Lin}(\{u, v\})$.

(36) $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2)$.

(37) $\dim(W_1 \cap W_2) \geq (\dim(W_1) + \dim(W_2)) - \dim(V)$.

(38) If V is the direct sum of W_1 and W_2, then $\dim(V) = \dim(W_1) + \dim(W_2)$.

5. The Fixed-Dimensional Subspace Family and the Pencil of Subspaces

One can prove the following proposition

(39) $n \leq \dim(V)$ iff there exists a strict subspace W of V such that $\dim(W) = n$.

Let G_1 be a field, let V be a finite dimensional vector space over G_1, and let n be a natural number. The functor $\mathrm{Sub}_n(V)$ yields a set and is defined as follows:

(Def.3) $x \in \mathrm{Sub}_n(V)$ iff there exists a strict subspace W of V such that $W = x$ and $\dim(W) = n$.

We now state three propositions:

(40) If $n \leq \dim(V)$, then $\mathrm{Sub}_n(V)$ is non empty.

(41) If $\dim(V) < n$, then $\mathrm{Sub}_n(V) = \emptyset$.

(42) $\mathrm{Sub}_n(W) \subseteq \mathrm{Sub}_n(V)$.

Let G_1 be a field, let V be a finite dimensional vector space over G_1, let W_2 be a subspace of V, and let W_1 be a strict subspace of W_2. Let us assume that $\dim(W_2) = \dim(W_1) + 2$. The functor $\mathbf{p}(W_1, W_2)$ yields a non empty set and is defined by:

(Def.4) $x \in \mathbf{p}(W_1, W_2)$ iff there exists a strict subspace W of W_2 such that $W = x$ and $\dim(W) = \dim(W_1) + 1$ and W_1 is a subspace of W.

We now state two propositions:

(43) Let W_1 be a strict subspace of W_2. Suppose $\dim(W_2) = \dim(W_1) + 2$. Then $x \in \mathbf{p}(W_1, W_2)$ if and only if there exists a strict subspace W of V such that $W = x$ and $\dim(W) = \dim(W_1) + 1$ and W_1 is a subspace of W and W is a subspace of W_2.

(44) For every strict subspace W_1 of W_2 such that $\dim(W_2) = \dim(W_1) + 2$ holds $\mathbf{p}(W_1, W_2) \subseteq \mathrm{Sub}_{\dim(W_1)+1}(V)$.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[7] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[9] Jarosław Kotowicz. Functions and finite sequences of real numbers. *Formalized Mathematics*, 3(**2**):275–278, 1992.

[10] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[11] Robert Milewski. Associated matrix of linear map. *Formalized Mathematics*, 5(**3**):339–345, 1996.

[12] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.

[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[14] Wojciech A. Trybulec. Basis of vector space. *Formalized Mathematics*, 1(**5**):883–885, 1990.

[15] Wojciech A. Trybulec. Linear combinations in real linear space. *Formalized Mathematics*, 1(**3**):581–588, 1990.

[16] Wojciech A. Trybulec. Linear combinations in vector space. *Formalized Mathematics*, 1(**5**):877–882, 1990.

[17] Wojciech A. Trybulec. Non-contiguous substrings and one-to-one finite sequences. *Formalized Mathematics*, 1(**3**):569–573, 1990.

[18] Wojciech A. Trybulec. Operations on subspaces in vector space. *Formalized Mathematics*, 1(**5**):871–876, 1990.

[19] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.

[20] Wojciech A. Trybulec. Subspaces and cosets of subspaces in vector space. *Formalized Mathematics*, 1(**5**):865–870, 1990.

[21] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[22] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[23] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[24] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received October 6, 1995