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Maximal Kolmogorov Subspaces of a Topological Space as Stone Retracts of the Ambient Space [1]

Zbigniew Karno
Warsaw University
Białystok

Summary. Let X be a topological space. X is said to be T_0-*space* (or *Kolmogorov space*) provided for every pair of distinct points x, $y \in X$ there exists an open subset of X containing exactly one of these points (see [1], [8], [4]). Such spaces and subspaces were investigated in Mizar formalism in [7]. A Kolmogorov subspace X_0 of a topological space X is said to be *maximal* provided for every Kolmogorov subspace Y of X if X_0 is subspace of Y then the topological structures of Y and X_0 are the same.

M.H. Stone proved in [10] that every topological space can be made into a Kolmogorov space by identifying points with the same closure (see also [11]). The purpose is to generalize the Stone result, using Mizar System. It is shown here that: (1) *in every topological space X there exists a maximal Kolmogorov subspace X_0 of X*, and (2) *every maximal Kolmogorov subspace X_0 of X is a continuous retract of X*. Moreover, *if $r : X \to X_0$ is a continuous retraction of X onto a maximal Kolmogorov subspace X_0 of X, then $r^{-1}(x) = \mathrm{MaxADSet}(x)$ for any point x of X belonging to X_0, where $\mathrm{MaxADSet}(x)$ is a unique maximal anti-discrete subset of X containing x* (see [5] for the precise definition of the set $\mathrm{MaxADSet}(x)$). The retraction r from the last theorem is defined uniquely, and it is denoted in the text by „Stone-retraction". It has the following two remarkable properties: r is open, i.e., sends open sets in X to open sets in X_0, and r is closed, i.e., sends closed sets in X to closed sets in X_0.

These results may be obtained by the methods described by R.H. Warren in [17].

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The terminology and notation used here are introduced in the following articles: [15], [16], [12], [18], [2], [3], [14], [9], [19], [13], [6], [5], and [7].

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1. Maximal T_0-Subsets

Let X be a topological space. Let us observe that a subset of X is T_0 if:

(Def.1) For all points a, b of X such that $a \in$ it and $b \in$ it holds if $a \neq b$, then $\mathrm{MaxADSet}(a) \cap \mathrm{MaxADSet}(b) = \emptyset$.

Let X be a topological space. Let us observe that a subset of X is T_0 if:

(Def.2) For every point a of X such that $a \in$ it holds it $\cap \mathrm{MaxADSet}(a) = \{a\}$.

Let X be a topological space. Let us observe that a subset of X is T_0 if:

(Def.3) For every point a of X such that $a \in$ it there exists a subset D of X such that D is maximal anti-discrete and it $\cap D = \{a\}$.

Let Y be a topological structure. A subset of Y is maximal T_0 if:

(Def.4) It is T_0 and for every subset D of Y such that D is T_0 and it $\subseteq D$ holds it $= D$.

Next we state the proposition

(1) Let Y_0, Y_1 be topological structures, and let D_0 be a subset of Y_0, and let D_1 be a subset of Y_1. Suppose the topological structure of $Y_0 =$ the topological structure of Y_1 and $D_0 = D_1$. If D_0 is maximal T_0, then D_1 is maximal T_0.

Let X be a topological space. Let us observe that a subset of X is maximal T_0 if:

(Def.5) It is T_0 and $\mathrm{MaxADSet}(\text{it}) =$ the carrier of X.

In the sequel X denotes a topological space.

We now state several propositions:

(2) For every subset M of X such that M is maximal T_0 holds M is dense.

(3) For every subset A of X such that A is open or closed holds if A is maximal T_0, then A is not proper.

(4) Every empty subset of X is not maximal T_0.

(5) Let M be a subset of X. Suppose M is maximal T_0. Let A be a subset of X. If A is closed, then $A = \mathrm{MaxADSet}(M \cap A)$.

(6) Let M be a subset of X. Suppose M is maximal T_0. Let A be a subset of X. If A is open, then $A = \mathrm{MaxADSet}(M \cap A)$.

(7) For every subset M of X such that M is maximal T_0 and for every subset A of X holds $\overline{A} = \mathrm{MaxADSet}(M \cap \overline{A})$.

(8) For every subset M of X such that M is maximal T_0 and for every subset A of X holds $\mathrm{Int}\, A = \mathrm{MaxADSet}(M \cap \mathrm{Int}\, A)$.

Let X be a topological space. Let us observe that a subset of X is maximal T_0 if:

(Def.6) For every point x of X there exists a point a of X such that $a \in$ it and it $\cap \mathrm{MaxADSet}(x) = \{a\}$.

The following two propositions are true:

(9) For every subset A of X such that A is T_0 there exists a subset M of X such that $A \subseteq M$ and M is maximal T_0.

(10) There exists subset of X which is maximal T_0.

2. Maximal Kolmogorov Subspaces

Let Y be a non empty topological structure. A subspace of Y is maximal T_0 if:

(Def.7) For every subset A of Y such that $A =$ the carrier of it holds A is maximal T_0.

One can prove the following proposition

(11) Let Y be a non empty topological structure, and let Y_0 be a subspace of Y, and let A be a subset of Y. Suppose $A =$ the carrier of Y_0. Then A is maximal T_0 if and only if Y_0 is maximal T_0.

Let Y be a non empty topological structure. Note that every subspace of Y which is maximal T_0 is also T_0 and every subspace of Y which is non T_0 is also non maximal T_0.

Let X be a topological space. Let us observe that a subspace of X is maximal T_0 if it satisfies the conditions (Def.8).

(Def.8) (i) It is T_0, and

(ii) for every T_0 subspace Y_0 of X such that it is a subspace of Y_0 holds the topological structure of it $=$ the topological structure of Y_0.

In the sequel X will be a topological space.

One can prove the following proposition

(12) Let A_0 be a non empty subset of X. Suppose A_0 is maximal T_0. Then there exists a strict subspace X_0 of X such that X_0 is maximal T_0 and $A_0 =$ the carrier of X_0.

Let X be a topological space. One can verify the following observations:

* every subspace of X which is maximal T_0 is also dense,
* every subspace of X which is non dense is also non maximal T_0,
* every subspace of X which is open and maximal T_0 is also non proper,
* every subspace of X which is open and proper is also non maximal T_0,
* every subspace of X which is proper and maximal T_0 is also non open,
* every subspace of X which is closed and maximal T_0 is also non proper,
* every subspace of X which is closed and proper is also non maximal T_0, and
* every subspace of X which is proper and maximal T_0 is also non closed.

Next we state the proposition

(13) Let Y_0 be a T_0 subspace of X. Then there exists a strict subspace X_0 of X such that Y_0 is a subspace of X_0 and X_0 is maximal T_0.

Let X be a topological space. Note that there exists a subspace of X which is maximal T_0 and strict.

Let X be a topological space. A maximal Kolmogorov subspace of X is a maximal T_0 subspace of X.

The following four propositions are true:

(14) Let X_0 be a maximal Kolmogorov subspace of X, and let G be a subset of X, and let G_0 be a subset of X_0. Suppose $G_0 = G$. Then G_0 is open if and only if the following conditions are satisfied:
  (i) MaxADSet(G) is open, and
  (ii) $G_0 = \mathrm{MaxADSet}(G) \cap$ (the carrier of X_0).

(15) Let X_0 be a maximal Kolmogorov subspace of X and let G be a subset of X. Then G is open if and only if the following conditions are satisfied:
  (i) $G = \mathrm{MaxADSet}(G)$, and
  (ii) there exists a subset G_0 of X_0 such that G_0 is open and $G_0 = G \cap$ (the carrier of X_0).

(16) Let X_0 be a maximal Kolmogorov subspace of X, and let F be a subset of X, and let F_0 be a subset of X_0. Suppose $F_0 = F$. Then F_0 is closed if and only if the following conditions are satisfied:
  (i) MaxADSet(F) is closed, and
  (ii) $F_0 = \mathrm{MaxADSet}(F) \cap$ (the carrier of X_0).

(17) Let X_0 be a maximal Kolmogorov subspace of X and let F be a subset of X. Then F is closed if and only if the following conditions are satisfied:
  (i) $F = \mathrm{MaxADSet}(F)$, and
  (ii) there exists a subset F_0 of X_0 such that F_0 is closed and $F_0 = F \cap$ (the carrier of X_0).

3. Stone Retraction Mapping Theorems

In the sequel X is a topological space and X_0 is a maximal Kolmogorov subspace of X.

One can prove the following propositions:

(18) Let r be a mapping from X into X_0 and let M be a subset of X. Suppose $M =$ the carrier of X_0. Suppose that for every point a of X holds $M \cap \mathrm{MaxADSet}(a) = \{r(a)\}$. Then r is a continuous mapping from X into X_0.

(19) Let r be a mapping from X into X_0. Suppose that for every point a of X holds $r(a) \in \mathrm{MaxADSet}(a)$. Then r is a continuous mapping from X into X_0.

(20) Let r be a continuous mapping from X into X_0 and let M be a subset of X. Suppose $M =$ the carrier of X_0. If for every point a of X holds $M \cap \mathrm{MaxADSet}(a) = \{r(a)\}$, then r is a retraction.

(21) For every continuous mapping r from X into X_0 such that for every point a of X holds $r(a) \in \mathrm{MaxADSet}(a)$ holds r is a retraction.

(22) There exists continuous mapping from X into X_0 which is a retraction.

(23) X_0 is a retract of X.

Let X be a topological space and let X_0 be a maximal Kolmogorov subspace of X. Stone-retraction of X onto X_0 is a continuous mapping from X into X_0 and is defined as follows:

(Def.9) Stone-retraction of X onto X_0 is a retraction.

Next we state three propositions:

(24) Let a be a point of X and let b be a point of X_0. If $a = b$, then (Stone-retraction of X onto X_0)$^{-1}\,\overline{\{b\}} = \overline{\{a\}}$.

(25) For every point a of X and for every point b of X_0 such that $a = b$ holds (Stone-retraction of X onto X_0)$^{-1}\,\{b\} = \mathrm{MaxADSet}(a)$.

(26) For every subset E of X and for every subset F of X_0 such that $F = E$ holds (Stone-retraction of X onto X_0)$^{-1}\,F = \mathrm{MaxADSet}(E)$.

Let X be a topological space and let X_0 be a maximal Kolmogorov subspace of X. Then Stone-retraction of X onto X_0 is a continuous mapping from X into X_0 and it can be characterized by the condition:

(Def.10) Let M be a subset of X. Suppose $M =$ the carrier of X_0. Let a be a point of X. Then $M \cap \mathrm{MaxADSet}(a) = \{$(Stone-retraction of X onto X_0)$(a)\}$.

Let X be a topological space and let X_0 be a maximal Kolmogorov subspace of X. Then Stone-retraction of X onto X_0 is a continuous mapping from X into X_0 and it can be characterized by the condition:

(Def.11) For every point a of X holds (Stone-retraction of X onto X_0)$(a) \in \mathrm{MaxADSet}(a)$.

Next we state two propositions:

(27) For every point a of X holds (Stone-retraction of X onto X_0)$^{-1}$ $\{$(Stone-retraction of X onto X_0)$(a)\} = \mathrm{MaxADSet}(a)$.

(28) For every point a of X holds (Stone-retraction of X onto X_0)$^{\circ}\{a\} =$ (Stone-retraction of X onto X_0)$^{\circ}\,\mathrm{MaxADSet}(a)$.

Let X be a topological space and let X_0 be a maximal Kolmogorov subspace of X. Then Stone-retraction of X onto X_0 is a continuous mapping from X into X_0 and it can be characterized by the condition:

(Def.12) Let M be a subset of X. Suppose $M =$ the carrier of X_0. Let A be a subset of X. Then $M \cap \mathrm{MaxADSet}(A) =$ (Stone-retraction of X onto X_0)$^{\circ}A$.

The following propositions are true:

(29) For every subset A of X holds (Stone-retraction of X onto X_0)$^{-1}$ (Stone-retraction of X onto X_0)$^{\circ}A = \mathrm{MaxADSet}(A)$.

(30) For every subset A of X holds (Stone-retraction of X onto X_0)$^{\circ}A =$ (Stone-retraction of X onto X_0)$^{\circ}\,\mathrm{MaxADSet}(A)$.

(31) Let A, B be subsets of X. Then (Stone-retraction of X onto X_0)$^\circ(A \cup B)$ = (Stone-retraction of X onto X_0)$^\circ A \cup$ (Stone-retraction of X onto X_0)$^\circ B$.

(32) Let A, B be subsets of X. Suppose A is open or B is open. Then (Stone-retraction of X onto X_0)$^\circ(A \cap B)$ = (Stone-retraction of X onto X_0)$^\circ A \cap$ (Stone-retraction of X onto X_0)$^\circ B$.

(33) Let A, B be subsets of X. Suppose A is closed or B is closed. Then (Stone-retraction of X onto X_0)$^\circ(A \cap B)$ = (Stone-retraction of X onto X_0)$^\circ A \cap$ (Stone-retraction of X onto X_0)$^\circ B$.

(34) For every subset A of X such that A is open holds (Stone-retraction of X onto X_0)$^\circ A$ is open.

(35) For every subset A of X such that A is closed holds (Stone-retraction of X onto X_0)$^\circ A$ is closed.

References

[1] P. Alexandroff and H. H. Hopf. *Topologie I.* Springer-Verlag, Berlin, 1935.
[2] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[3] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[4] Ryszard Engelking. *General Topology.* Volume 60 of *Monografie Matematyczne*, PWN - Polish Scientific Publishers, Warsaw, 1977.
[5] Zbigniew Karno. Maximal anti-discrete subspaces of topological spaces. *Formalized Mathematics*, 5(**1**):109–118, 1996.
[6] Zbigniew Karno. Maximal discrete subspaces of almost discrete topological spaces. *Formalized Mathematics*, 4(**1**):125–135, 1993.
[7] Zbigniew Karno. On Kolmogorov topological spaces. *Formalized Mathematics*, 5(**1**):119–124, 1996.
[8] Kazimierz Kuratowski. *Topology.* Volume I, PWN - Polish Scientific Publishers, Academic Press, Warsaw, New York and London, 1966.
[9] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[10] M. H. Stone. Application of boolean algebras to topology. *Math. Sb.*, 1:765–771, 1936.
[11] W.J. Thron. *Topological Structures.* Holt, Rinehart and Winston, New York, 1966.
[12] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[13] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Formalized Mathematics*, 2(**4**):535–545, 1991.
[14] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[16] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[17] R.H. Warren. Identification spaces and unique uniformity. *Pacific Journal of Mathematics*, 95:483–492, 1981.
[18] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[19] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Formalized Mathematics*, 1(**1**):231–237, 1990.

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