FORMALIZED MATHEMATICS
Volume 5, Number 1, 1996
Warsaw University - Białystok

The Formalization of Simple Graphs

Yozo Toda
Information Processing Center
Chiba University

Summary. A graph is simple when

- it is non-directed,
- there is at most one edge between two vertices,
- there is no loop of length one.

A formalization of simple graphs is given from scratch. There is already an article [9], dealing with the similar subject. It is not used as a starting-point, because [9] formalizes directed non-empty graphs. Given a set of vertices, edge is defined as an (unordered) pair of different two vertices and graph as a pair of a set of vertices and a set of edges.

The following concepts are introduced:

- simple graph structure,
- the set of all simple graphs,
- equality relation on graphs.
- the notion of degrees of vertices; the number of edges connected to, or the number of adjacent vertices,
- the notion of subgraphs,
- path, cycle,
- complete and bipartite complete graphs,

Theorems proved in this articles include:

- the set of simple graphs satisfies a certain minimality condition,
- equivalence between two notions of degrees.

MML Identifier: `SGRAPH1`.

The terminology and notation used in this paper have been introduced in the following articles: [13], [1], [4], [6], [7], [2], [3], [8], [5], [11], [10], and [12].

ISSN 0777–4028

1. Preliminaries

Let m, n be natural numbers. The functor $[m,n]_{\mathbb{N}}$ yields a finite subset of $\mathbb{N}$ and is defined by:

(Def.1) $[m,n]_{\mathbb{N}} = \{i : i$ ranges over natural numbers, $m \leq i \ \wedge \ i \leq n\}$.

The following propositions are true:

(1) For all natural numbers m, n holds $[m,n]_{\mathbb{N}} = \{i : i$ ranges over natural numbers, $m \leq i \ \wedge \ i \leq n\}$.

(2) Let m, n be natural numbers and let e be arbitrary. Then $e \in [m,n]_{\mathbb{N}}$ if and only if there exists a natural number i such that $e = i$ and $m \leq i$ and $i \leq n$.

(3) For all natural numbers m, n, k holds $k \in [m,n]_{\mathbb{N}}$ iff $m \leq k$ and $k \leq n$.

(4) For every natural number n holds $[1,n]_{\mathbb{N}} = \operatorname{Seg} n$.

(5) For all natural numbers m, n such that $1 \leq m$ holds $[m,n]_{\mathbb{N}} \subseteq \operatorname{Seg} n$.

(6) For all natural numbers k, m, n such that $k < m$ holds $\operatorname{Seg} k \cap [m,n]_{\mathbb{N}} = \emptyset$.

(7) For all natural numbers m, n such that $n < m$ holds $[m,n]_{\mathbb{N}} = \emptyset$.

Let A, B be sets and let f be a function from A into B. We say that f is onto if and only if:

(Def.2) $\operatorname{rng} f = B$.

Let A, B be sets and let f be a function from A into B. We say that f is bijective if and only if:

(Def.3) f is one-to-one and onto.

One can prove the following proposition

(8) For every finite set z holds $\operatorname{card} z = 2$ iff there exist arbitrary x, y such that $x \in z$ and $y \in z$ and $x \neq y$ and $z = \{x, y\}$.

Let A be a set. The functor TwoElementSets(A) yields a set and is defined by:

(Def.4) $\text{TwoElementSets}(A) = \{z : z$ ranges over finite elements of 2^A, $\operatorname{card} z = 2\}$.

The following propositions are true:

(9) For every set A and for arbitrary e holds $e \in \text{TwoElementSets}(A)$ iff there exists a finite subset z of A such that $e = z$ and $\operatorname{card} z = 2$.

(10) Let A be a set and let e be arbitrary. Then $e \in \text{TwoElementSets}(A)$ if and only if the following conditions are satisfied:

(i) e is a finite subset of A, and

(ii) there exist arbitrary x, y such that $x \in A$ and $y \in A$ and $x \neq y$ and $e = \{x, y\}$.

(11) For every set A holds $\text{TwoElementSets}(A) \subseteq 2^A$.

(12) For every set A and for arbitrary e_1, e_2 such that $\{e_1, e_2\} \in$ TwoElementSets(A) holds $e_1 \in A$ and $e_2 \in A$ and $e_1 \neq e_2$.

(13) TwoElementSets$(\emptyset) = \emptyset$.

(14) For all sets t, u such that $t \subseteq u$ holds TwoElementSets$(t) \subseteq$ TwoElementSets(u).

(15) For every finite set A holds TwoElementSets(A) is finite.

(16) For every non trivial set A holds TwoElementSets(A) is non empty.

(17) For arbitrary a holds TwoElementSets$(\{a\}) = \emptyset$.

Let a be a set.

(Def.5) $\phi(a)$ is an empty subset of TwoElementSets(a).

Let X be an empty set. Observe that every subset of X is empty.

In the sequel X will be a set.

2. Simple Graphs

We introduce simple graph structures which are systems

⟨ SVertices, SEdges ⟩,

where the SVertices constitute a set and the SEdges constitute a subset of TwoElementSets(the SVertices).

Let X be a set. The functor SimpleGraphs(X) yields a non empty set and is defined as follows:

(Def.6) SimpleGraphs$(X) = \{\langle v, e\rangle : v$ ranges over finite subsets of X, e ranges over finite subsets of TwoElementSets$(v)\}$.

Next we state the proposition

(19)[1] $\langle \emptyset, \phi(\emptyset)\rangle \in$ SimpleGraphs(X).

Let X be a set. A strict simple graph structure is said to be a simple graph of X if:

(Def.7) It is an element of SimpleGraphs(X).

Next we state two propositions:

(20) SimpleGraphs$(X) = \{\langle v, e\rangle : v$ ranges over finite subsets of X, e ranges over finite subsets of TwoElementSets$(v)\}$.

(21) Let g be arbitrary. Then $g \in$ SimpleGraphs(X) if and only if there exists a finite subset v of X and there exists a finite subset e of TwoElementSets(v) such that $g = \langle v, e\rangle$.

[1]The proposition (18) has been removed.

3. Equality Relation on Simple Graphs

One can prove the following propositions:

(23)[2] For every simple graph g of X holds the SVertices of $g \subseteq X$ and the SEdges of $g \subseteq$ TwoElementSets(the SVertices of g).

(24) For every simple graph g of X holds $g = \langle$the SVertices of g, the SEdges of $g\rangle$.

(25) Let g be a simple graph of X and let e be arbitrary. Suppose $e \in$ the SEdges of g. Then there exist arbitrary v_1, v_2 such that $v_1 \in$ the SVertices of g and $v_2 \in$ the SVertices of g and $v_1 \neq v_2$ and $e = \{v_1, v_2\}$.

(26) Let g be a simple graph of X and let v_1, v_2 be arbitrary. Suppose $\{v_1, v_2\} \in$ the SEdges of g. Then $v_1 \in$ the SVertices of g and $v_2 \in$ the SVertices of g and $v_1 \neq v_2$.

(27) Let g be a simple graph of X. Then

(i) the SVertices of g is a finite subset of X, and

(ii) the SEdges of g is a finite subset of TwoElementSets(the SVertices of g).

Let us consider X and let G, G' be simple graphs of X. We say that G is isomorphic to G' if and only if the condition (Def.8) is satisfied.

(Def.8) There exists a function F_1 from the SVertices of G into the SVertices of G' such that

(i) F_1 is bijective, and

(ii) for all elements v_1, v_2 of the SVertices of G holds $\{v_1, v_2\} \in$ the SEdges of G iff $\{F_1(v_1), F_1(v_2)\} \in$ the SEdges of G.

4. Properties of Simple Graphs

The scheme *IndSimpleGraphs0* concerns a set $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For arbitrary G such that $G \in$ SimpleGraphs($\mathcal{A}$) holds $\mathcal{P}[G]$

provided the parameters satisfy the following conditions:

- $\mathcal{P}[\langle\emptyset, \phi(\emptyset)\rangle]$,
- Let g be a simple graph of $\mathcal{A}$ and let v be arbitrary. Suppose $g \in$ SimpleGraphs($\mathcal{A}$) and $\mathcal{P}[g]$ and $v \in \mathcal{A}$ and $v \notin$ the SVertices of g. Then $\mathcal{P}[\langle(\text{the SVertices of } g) \cup \{v\}, \phi((\text{the SVertices of } g) \cup \{v\})\rangle]$,
- Let g be a simple graph of $\mathcal{A}$ and let e be arbitrary. Suppose $\mathcal{P}[g]$ and $e \in$ TwoElementSets(the SVertices of g) and $e \notin$ the SEdges of g. Then there exists a subset s_1 of TwoElementSets(the SVertices of g) such that $s_1 =$ (the SEdges of g) $\cup \{e\}$ and $\mathcal{P}[\langle$the SVertices of g, $s_1\rangle]$.

[2]The proposition (22) has been removed.

We now state three propositions:

(28) Let g be a simple graph of X. Then $g = \langle \emptyset, \phi(\emptyset) \rangle$ or there exists a set v and there exists a subset e of TwoElementSets(v) such that v is non empty and $g = \langle v, e \rangle$.

(30)[3] Let V be a subset of X, and let E be a subset of TwoElementSets(V), and let n be arbitrary, and let E_1 be a finite subset of TwoElementSets($V \cup \{n\}$). If $\langle V, E \rangle \in$ SimpleGraphs(X) and $n \in X$ and $n \notin V$, then $\langle V \cup \{n\}, E_1 \rangle \in$ SimpleGraphs(X).

(31) Let V be a subset of X, and let E be a subset of TwoElementSets(V), and let v_1, v_2 be arbitrary. Suppose $v_1 \in V$ and $v_2 \in V$ and $v_1 \neq v_2$ and $\langle V, E \rangle \in$ SimpleGraphs(X). Then there exists a finite subset v_3 of TwoElementSets(V) such that $v_3 = E \cup \{\{v_1, v_2\}\}$ and $\langle V, v_3 \rangle \in$ SimpleGraphs(X).

Let X be a set and let G_1 be a set. We say that G_1 is a set of simple graphs of X if and only if the conditions (Def.9) are satisfied.

(Def.9) (i) $\langle \emptyset, \phi(\emptyset) \rangle \in G_1$,

(ii) for every subset V of X and for every subset E of TwoElementSets(V) and for arbitrary n and for every finite subset E_1 of TwoElementSets($V \cup \{n\}$) such that $\langle V, E \rangle \in G_1$ and $n \in X$ and $n \notin V$ holds $\langle V \cup \{n\}, E_1 \rangle \in G_1$, and

(iii) for every subset V of X and for every subset E of TwoElementSets(V) and for arbitrary v_1, v_2 such that $\langle V, E \rangle \in G_1$ and $v_1 \in V$ and $v_2 \in V$ and $v_1 \neq v_2$ and $\{v_1, v_2\} \notin E$ there exists a finite subset v_3 of TwoElementSets(V) such that $v_3 = E \cup \{\{v_1, v_2\}\}$ and $\langle V, v_3 \rangle \in G_1$.

One can prove the following propositions:

(32) For arbitrary g_1 such that g_1 is a set of simple graphs of X holds $\langle \emptyset, \phi(\emptyset) \rangle \in g_1$.

(33) Let G_1 be arbitrary. Suppose G_1 is a set of simple graphs of X. Let V be a subset of X, and let E be a subset of TwoElementSets(V), and let n be arbitrary, and let E_1 be a finite subset of TwoElementSets($V \cup \{n\}$). If $\langle V, E \rangle \in G_1$ and $n \in X$ and $n \notin V$, then $\langle V \cup \{n\}, E_1 \rangle \in G_1$.

(34) Let G_1 be arbitrary. Suppose G_1 is a set of simple graphs of X. Let V be a subset of X, and let E be a subset of TwoElementSets(V), and let v_1, v_2 be arbitrary. Suppose $\langle V, E \rangle \in G_1$ and $v_1 \in V$ and $v_2 \in V$ and $v_1 \neq v_2$ and $\{v_1, v_2\} \notin E$. Then there exists a finite subset v_3 of TwoElementSets(V) such that $v_3 = E \cup \{\{v_1, v_2\}\}$ and $\langle V, v_3 \rangle \in G_1$.

(35) SimpleGraphs(X) is a set of simple graphs of X.

(36) For arbitrary O_1 such that O_1 is a set of simple graphs of X holds SimpleGraphs(X) $\subseteq O_1$.

(37) SimpleGraphs(X) is a set of simple graphs of X and for arbitrary O_1 such that O_1 is a set of simple graphs of X holds SimpleGraphs(X) $\subseteq O_1$.

[3]The proposition (29) has been removed.

5. Subgraphs

Let X be a set and let G be a simple graph of X. A simple graph of X is called a subgraph of G if:

(Def.10) The SVertices of it $\subseteq$ the SVertices of G and the SEdges of it $\subseteq$ the SEdges of G.

6. Degree of Vertices

Let X be a set, let G be a simple graph of X, and let v be arbitrary. Let us assume that $v \in$ the SVertices of G. The functor degree(G, v) yielding a natural number is defined by:

(Def.11) There exists a finite set X such that for arbitrary z holds $z \in X$ iff $z \in$ the SEdges of G and $v \in z$ and degree$(G, v) = \operatorname{card} X$.

One can prove the following propositions:

(38) Let G be a simple graph of X and let v be arbitrary. Suppose $v \in$ the SVertices of G. Then there exists a finite set Y such that for arbitrary z holds $z \in Y$ iff $z \in$ the SEdges of G and $v \in z$ and degree$(G, v) = \operatorname{card} Y$.

(39) Let X be a non empty set, and let G be a simple graph of X, and let v be arbitrary. Suppose $v \in$ the SVertices of G. Then there exists a finite set w_1 such that $w_1 = \{w : w$ ranges over elements of X, $w \in$ the SVertices of $G \wedge \{v, w\} \in$ the SEdges of $G\}$ and degree$(G, v) = \operatorname{card} w_1$.

(40) Let X be a non empty set, and let g be a simple graph of X, and let v be arbitrary. Suppose $v \in$ the SVertices of g. Then there exists a finite set V_1 such that $V_1 =$ the SVertices of g and degree$(g, v) < \operatorname{card} V_1$.

(41) Let g be a simple graph of X and let v, e be arbitrary. Suppose $v \in$ the SVertices of g and $e \in$ the SEdges of g and degree$(g, v) = 0$. Then $v \notin e$.

(42) Let g be a simple graph of X, and let v be arbitrary, and let v_4 be a finite set. Suppose $v_4 =$ the SVertices of g and $v \in v_4$ and $1 +$ degree$(g, v) = \operatorname{card} v_4$. Let w be an element of v_4. If $v \neq w$, then there exists arbitrary e such that $e \in$ the SEdges of g and $e = \{v, w\}$.

7. Path and Cycle

Let X be a set, let g be a simple graph of X, let v_1, v_2 be elements of the SVertices of g, and let p be a finite sequence of elements of the SVertices of g. We say that p is a path of v_1 and v_2 if and only if the conditions (Def.12) are satisfied.

(Def.12) (i) $p(1) = v_1$,
(ii) $p(\mathrm{len}\, p) = v_2$,
(iii) for every natural number i such that $1 \leq i$ and $i < \mathrm{len}\, p$ holds $\{p(i), p(i+1)\} \in$ the SEdges of g, and
(iv) for all natural numbers i, j such that $1 \leq i$ and $i < \mathrm{len}\, p$ and $i < j$ and $j < \mathrm{len}\, p$ holds $p(i) \neq p(j)$ and $\{p(i), p(i+1)\} \neq \{p(j), p(j+1)\}$.

Let X be a set, let g be a simple graph of X, and let v_1, v_2 be elements of the SVertices of g. The functor $\mathrm{Paths}(v_1, v_2)$ yields a subset of (the SVertices of g)* and is defined by:

(Def.13) $\mathrm{Paths}(v_1, v_2) = \{s_2 :$ s_2 ranges over elements of (the SVertices of g)*, s_2 is a path of v_1 and $v_2\}$.

One can prove the following three propositions:

(43) Let g be a simple graph of X and let v_1, v_2 be elements of the SVertices of g. Then $\mathrm{Paths}(v_1, v_2) = \{s_2 :$ s_2 ranges over elements of (the SVertices of g)*, s_2 is a path of v_1 and $v_2\}$.

(44) Let g be a simple graph of X, and let v_1, v_2 be elements of the SVertices of g, and let e be arbitrary. Then $e \in \mathrm{Paths}(v_1, v_2)$ if and only if there exists an element s_2 of (the SVertices of g)* such that $e = s_2$ and s_2 is a path of v_1 and v_2.

(45) Let g be a simple graph of X, and let v_1, v_2 be elements of the SVertices of g, and let e be an element of (the SVertices of g)*. If e is a path of v_1 and v_2, then $e \in \mathrm{Paths}(v_1, v_2)$.

Let X be a set, let g be a simple graph of X, and let p be arbitrary. We say that p is a cycle of g if and only if:

(Def.14) There exists an element v of the SVertices of g such that $p \in \mathrm{Paths}(v, v)$.

8. Some Famous Graphs

Let n, m be natural numbers. The functor $\mathrm{K}_{m,n}$ yielding a simple graph of $\mathbb{N}$ is defined by the condition (Def.16).

(Def.16)[4] There exists a subset e_3 of TwoElementSets(Seg($m+n$)) such that $e_3 = \{\{i, j\} :$ i ranges over elements of $\mathbb{N}$, j ranges over elements of $\mathbb{N}$, $i \in \mathrm{Seg}\, m \ \wedge\ j \in [m+1, m+n]_{\mathbb{N}}\}$ and $\mathrm{K}_{m,n} = \langle \mathrm{Seg}(m+n), e_3 \rangle$.

Let n be a natural number. The functor K_n yields a simple graph of $\mathbb{N}$ and is defined by the condition (Def.17).

(Def.17) There exists a finite subset e_3 of TwoElementSets(Seg n) such that $e_3 = \{\{i, j\} :$ i ranges over elements of $\mathbb{N}$, j ranges over elements of $\mathbb{N}$, $i \in \mathrm{Seg}\, n \ \wedge\ j \in \mathrm{Seg}\, n \ \wedge\ i \neq j\}$ and $\mathrm{K}_n = \langle \mathrm{Seg}\, n, e_3 \rangle$.

The simple graph TriangleGraph of $\mathbb{N}$ is defined by:

(Def.18) TriangleGraph $= \mathrm{K}_3$.

[4]The definition (Def.15) has been removed.

One can prove the following propositions:

(46) There exists a subset e_3 of TwoElementSets(Seg 3) such that $e_3 = \{\{1,2\},\{2,3\},\{3,1\}\}$ and TriangleGraph $= \langle \mathrm{Seg}\,3, e_3\rangle$.

(47) The SVertices of TriangleGraph $=$ Seg 3 and the SEdges of TriangleGraph $= \{\{1,2\},\{2,3\},\{3,1\}\}$.

(48) $\{1,2\} \in$ the SEdges of TriangleGraph and $\{2,3\} \in$ the SEdges of TriangleGraph and $\{3,1\} \in$ the SEdges of TriangleGraph.

(49) $\langle 1\rangle \frown \langle 2\rangle \frown \langle 3\rangle \frown \langle 1\rangle$ is a cycle of TriangleGraph.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[4] Józef Białas. Group and field definitions. *Formalized Mathematics*, 1(**3**):433–439, 1990.
[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[8] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[9] Krzysztof Hryniewiecki. Graphs. *Formalized Mathematics*, 2(**3**):365–370, 1991.
[10] Andrzej Trybulec. Semilattice operations on finite subsets. *Formalized Mathematics*, 1(**2**):369–376, 1990.
[11] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1(**1**):187–190, 1990.
[12] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[13] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

Received September 8, 1994