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Projective Planes

Michał Muzalewski
Warsaw University
Białystok

Summary. The line of points a, b, denoted by $a \cdot b$ and the point of lines A, B denoted by $A \cdot B$ are defined. A few basic theorems related to these notions are proved. An inspiration for such approach comes from so called Leibniz program. Let us recall that the Leibniz program is a program of algebraization of geometry using purely geometric notions. Leibniz formulated his program in opposition to algebraization method developed by Descartes.

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The terminology and notation used in this paper are introduced in the papers [2] and [1].

1. Projective Spaces

In this paper G will denote a projective incidence structure.

Let us consider G. A point of G is an element of the points of G. A line of G is an element of the lines of G.

We adopt the following rules: a, a_1, a_2, b, b_1, b_2, c, d, p, q, r will be points of G and A, B, M, N, P, Q, R will be lines of G.

Let us consider G, a, P. We introduce $a \nmid P$ as an antonym of $a \mid P$.

Let us consider G, a, b, P. The predicate $a, b \nmid P$ is defined as follows:

(Def.1) $a \nmid P$ and $b \nmid P$.

Let us consider G, a, P, Q. The predicate $a \mid P, Q$ is defined as follows:

(Def.2) $a \mid P$ and $a \mid Q$.

Let us consider G, a, P, Q, R. The predicate $a \mid P, Q, R$ is defined as follows:

(Def.3) $a \mid P$ and $a \mid Q$ and $a \mid R$.

We now state the proposition

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(1) (i) If $a, b \mid P$, then $b, a \mid P$,
(ii) if $a, b, c \mid P$, then $a, c, b \mid P$ and $b, a, c \mid P$ and $b, c, a \mid P$ and $c, a, b \mid P$ and $c, b, a \mid P$,
(iii) if $a \mid P, Q$, then $a \mid Q, P$, and
(iv) if $a \mid P, Q, R$, then $a \mid P, R, Q$ and $a \mid Q, P, R$ and $a \mid Q, R, P$ and $a \mid R, P, Q$ and $a \mid R, Q, P$.

A projective incidence structure is configuration if:

(Def.4) For all points p, q of it and for all lines P, Q of it such that $p \mid P$ and $q \mid P$ and $p \mid Q$ and $q \mid Q$ holds $p = q$ or $P = Q$.

We now state three propositions:

(2) G is configuration iff for all p, q, P, Q such that $p, q \mid P$ and $p, q \mid Q$ holds $p = q$ or $P = Q$.

(3) G is configuration iff for all p, q, P, Q such that $p \mid P, Q$ and $q \mid P, Q$ holds $p = q$ or $P = Q$.

(4) The following statements are equivalent
(i) G is a projective space defined in terms of incidence,
(ii) G is configuration and for all p, q there exists P such that $p, q \mid P$ and there exist p, P such that $p \nmid P$ and for every P there exist a, b, c such that a, b, c are mutually different and $a, b, c \mid P$ and for all a, b, c, d, p, M, N, P, Q such that $a, b, p \mid M$ and $c, d, p \mid N$ and $a, c \mid P$ and $b, d \mid Q$ and $p \nmid P$ and $p \nmid Q$ and $M \neq N$ there exists q such that $q \mid P, Q$.

An incidence projective plane is a 2-dimensional projective space defined in terms of incidence.

Let us consider G, a, b, c. We say that a, b and c are collinear if and only if:

(Def.5) There exists P such that $a, b, c \mid P$.

We introduce a, b, c form a triangle as an antonym of a, b and c are collinear.

Next we state two propositions:

(5) a, b and c are collinear iff there exists P such that $a \mid P$ and $b \mid P$ and $c \mid P$.

(6) a, b, c form a triangle iff for every P holds $a \nmid P$ or $b \nmid P$ or $c \nmid P$.

Let us consider G, a, b, c, d. We say that a, b, c, d form a quadrangle if and only if the conditions (Def.6) are satisfied.

(Def.6) (i) a, b, c form a triangle,
(ii) b, c, d form a triangle,
(iii) c, d, a form a triangle, and
(iv) d, a, b form a triangle.

One can prove the following propositions:

(7) If G is a projective space defined in terms of incidence, then there exist A, B such that $A \neq B$.

(8) Suppose G is a projective space defined in terms of incidence and $a \mid A$. Then there exist b, c such that $b, c \mid A$ and a, b, c are mutually different.

(9) Suppose G is a projective space defined in terms of incidence and $a \mid A$ and $A \neq B$. Then there exists b such that $b \mid A$ and $b \nmid B$ and $a \neq b$.

(10) If G is configuration and $a_1, a_2 \mid A$ and $a_1 \neq a_2$ and $b \nmid A$, then a_1, a_2, b form a triangle.

(11) Suppose a, b and c are collinear. Then
(i) a, c and b are collinear,
(ii) b, a and c are collinear,
(iii) b, c and a are collinear,
(iv) c, a and b are collinear, and
(v) c, b and a are collinear.

(12) Suppose a, b, c form a triangle. Then
(i) a, c, b form a triangle,
(ii) b, a, c form a triangle,
(iii) b, c, a form a triangle,
(iv) c, a, b form a triangle, and
(v) c, b, a form a triangle.

(13) Suppose a, b, c, d form a quadrangle. Then
(i) a, c, b, d form a quadrangle,
(ii) b, a, c, d form a quadrangle,
(iii) b, c, a, d form a quadrangle,
(iv) c, a, b, d form a quadrangle,
(v) c, b, a, d form a quadrangle,
(vi) a, b, d, c form a quadrangle,
(vii) a, c, d, b form a quadrangle,
(viii) b, a, d, c form a quadrangle,
(ix) b, c, d, a form a quadrangle,
(x) c, a, d, b form a quadrangle,
(xi) c, b, d, a form a quadrangle,
(xii) a, d, b, c form a quadrangle,
(xiii) a, d, c, b form a quadrangle,
(xiv) b, d, a, c form a quadrangle,
(xv) b, d, c, a form a quadrangle,
(xvi) c, d, a, b form a quadrangle,
(xvii) c, d, b, a form a quadrangle,
(xviii) d, a, b, c form a quadrangle,
(xix) d, a, c, b form a quadrangle,
(xx) d, b, a, c form a quadrangle,
(xxi) d, b, c, a form a quadrangle,
(xxii) d, c, a, b form a quadrangle, and
(xxiii) d, c, b, a form a quadrangle.

(14) If G is configuration and $a_1, a_2 \mid A$ and $b_1, b_2 \mid B$ and $a_1, a_2 \nmid B$ and $b_1, b_2 \nmid A$ and $a_1 \neq a_2$ and $b_1 \neq b_2$, then a_1, a_2, b_1, b_2 form a quadrangle.

(15) Suppose G is a projective space defined in terms of incidence. Then there exist a, b, c, d such that a, b, c, d form a quadrangle.

Let G be a projective space defined in terms of incidence. An element of [: the points of G, the points of G, the points of G, the points of G :] is called a quadrangle of G if:

(Def.7) $\mathrm{it}_{\mathbf{1}}$, $\mathrm{it}_{\mathbf{2}}$, $\mathrm{it}_{\mathbf{3}}$, $\mathrm{it}_{\mathbf{4}}$ form a quadrangle.

Let G be a projective space defined in terms of incidence and let a, b be points of G. Let us assume that $a \neq b$. The functor $a \cdot b$ yields a line of G and is defined as follows:

(Def.8) $a, b \mid a \cdot b$.

Next we state the proposition

(16) Let G be a projective space defined in terms of incidence, and let a, b be points of G, and let L be a line of G. Suppose $a \neq b$. Then $a \mid a \cdot b$ and $b \mid a \cdot b$ and $a \cdot b = b \cdot a$ and if $a \mid L$ and $b \mid L$, then $L = a \cdot b$.

2. Projective Planes

The following propositions are true:

(17) If there exist a, b, c such that a, b, c form a triangle and for all p, q there exists M such that $p, q \mid M$, then there exist p, P such that $p \nmid P$.

(18) If there exist a, b, c, d such that a, b, c, d form a quadrangle, then there exist a, b, c such that a, b, c form a triangle.

(19) If a, b, c form a triangle and $a, b \mid P$ and $a, c \mid Q$, then $P \neq Q$.

(20) If a, b, c, d form a quadrangle and $a, b \mid P$ and $a, c \mid Q$ and $a, d \mid R$, then P, Q, R are mutually different.

(21) Suppose G is configuration and $a \mid P, Q, R$ and P, Q, R are mutually different and $a \nmid A$ and $p \mid A, P$ and $q \mid A, Q$ and $r \mid A, R$. Then p, q, r are mutually different.

(22) Suppose that

(i) G is configuration,

(ii) for all p, q there exists M such that $p, q \mid M$,

(iii) for all P, Q there exists a such that $a \mid P, Q$, and

(iv) there exist a, b, c, d such that a, b, c, d form a quadrangle.

Given P. Then there exist a, b, c such that a, b, c are mutually different and $a, b, c \mid P$.

(23) G is an incidence projective plane if and only if the following conditions are satisfied:

(i) G is configuration,

(ii) for all p, q there exists M such that $p, q \mid M$,

(iii) for all P, Q there exists a such that $a \mid P, Q$, and

(iv) there exist a, b, c, d such that a, b, c, d form a quadrangle.

We adopt the following convention: G will denote an incidence projective plane, a, q will denote points of G, and A, B will denote lines of G.

Let us consider G, A, B. Let us assume that $A \neq B$. The functor $A \cdot B$ yields a point of G and is defined by:

(Def.9) $A \cdot B \mid A, B$.

Next we state two propositions:

(24) If $A \neq B$, then $A \cdot B \mid A$ and $A \cdot B \mid B$ and $A \cdot B = B \cdot A$ and if $a \mid A$ and $a \mid B$, then $a = A \cdot B$.

(25) If $A \neq B$ and $a \mid A$ and $q \nmid A$ and $a \neq A \cdot B$, then $q \cdot a \cdot B \mid B$ and $q \cdot a \cdot B \nmid A$.

3. Some Useful Propositions

We adopt the following convention: G denotes a projective space defined in terms of incidence and a, b, c, d denote points of G.

We now state two propositions:

(26) If a, b, c form a triangle, then a, b, c are mutually different.

(27) If a, b, c, d form a quadrangle, then a, b, c, d are mutually different.

In the sequel G will be an incidence projective plane.

One can prove the following propositions:

(28) For all points a, b, c, d of G such that $a \cdot c = b \cdot d$ holds $a = c$ or $b = d$ or $c = d$ or $a \cdot c = c \cdot d$.

(29) For all points a, b, c, d of G such that $a \cdot c = b \cdot d$ holds $a = c$ or $b = d$ or $c = d$ or $a \mid c \cdot d$.

(30) Let G be an incidence projective plane, and let e, m, m' be points of G, and let I be a line of G. If $m \mid I$ and $m' \mid I$ and $m \neq m'$ and $e \nmid I$, then $m \cdot e \neq m' \cdot e$ and $e \cdot m \neq e \cdot m'$.

(31) Let G be an incidence projective plane, and let e be a point of G, and let I, L_1, L_2 be lines of G. If $e \mid L_1$ and $e \mid L_2$ and $L_1 \neq L_2$ and $e \nmid I$, then $I \cdot L_1 \neq I \cdot L_2$ and $L_1 \cdot I \neq L_2 \cdot I$.

(32) Let G be a projective space defined in terms of incidence and let a, b, q, q_1 be points of G. If $q \mid a \cdot b$ and $q \mid a \cdot q_1$ and $q \neq a$ and $q_1 \neq a$ and $a \neq b$, then $q_1 \mid a \cdot b$.

(33) Let G be a projective space defined in terms of incidence and let a, b, c be points of G. If $c \mid a \cdot b$ and $a \neq c$, then $b \mid a \cdot c$.

(34) Let G be an incidence projective plane, and let q_1, q_2, r_1, r_2 be points of G, and let H be a line of G. If $r_1 \neq r_2$ and $r_1 \mid H$ and $r_2 \mid H$ and $q_1 \nmid H$ and $q_2 \nmid H$, then $q_1 \cdot r_1 \neq q_2 \cdot r_2$.

(35) For all points a, b, c of G such that $a \mid b \cdot c$ holds $a = c$ or $b = c$ or $b \mid c \cdot a$.

(36) For all points a, b, c of G such that $a \mid b \cdot c$ holds $b = a$ or $b = c$ or $c \mid b \cdot a$.

(37) Let e, x_1, x_2, p_1, p_2 be points of G and let H, I be lines of G. Suppose $x_1 \mid I$ and $x_2 \mid I$ and $e \mid H$ and $e \nmid I$ and $x_1 \neq x_2$ and $p_1 \neq e$ and $p_2 \neq e$ and $p_1 \mid e \cdot x_1$ and $p_2 \mid e \cdot x_2$. Then there exists a point r of G such that $r \mid p_1 \cdot p_2$ and $r \mid H$ and $r \neq e$.

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