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Maximal Discrete Subspaces of Almost Discrete Topological Spaces

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Summary. Let X be a topological space and let D be a subset of X. D is said to be *discrete* provided for every subset A of X such that $A \subseteq D$ there is an open subset G of X such that $A = D \cap G$ (comp. e.g., [7]). A discrete subset M of X is said to be *maximal discrete* provided for every discrete subset D of X if $M \subseteq D$ then $M = D$. A subspace of X is *discrete* (*maximal discrete*) iff its carrier is discrete (maximal discrete) in X.

Our purpose is to list a number of properties of discrete and maximal discrete sets in Mizar formalism. In particular, we show here that *if D is dense and discrete then D is maximal discrete*; moreover, *if D is open and maximal discrete then D is dense.* We discuss also the problem of the existence of maximal discrete subsets in a topological space.

To present the main results we first recall a definition of a class of topological spaces considered herein. A topological space X is called *almost discrete* if every open subset of X is closed; equivalently, if every closed subset of X is open. Such spaces were investigated in Mizar formalism in [4] and [5]. We show here that *every almost discrete space contains a maximal discrete subspace and every such subspace is a retract of the enveloping space.* Moreover, *if X_0 is a maximal discrete subspace of an almost discrete space X and $r : X \to X_0$ is a continuous retraction, then $r^{-1}(x) = \overline{\{x\}}$ for every point x of X belonging to X_0.* This fact is a specialization, in the case of almost discrete spaces, of the theorem of M.H. Stone that every topological space can be made into a T_0-space by suitable identification of points (see [9]).

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The terminology and notation used in this paper are introduced in the following papers: [13], [14], [10], [2], [3], [12], [1], [8], [15], [11], [4], and [6].

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1. Proper Subsets of 1-sorted Structures

A non empty set is trivial if:

(Def.1) There exists an element s of it such that it $= \{s\}$.

Let us note that there exists a non empty set which is trivial and there exists a non empty set which is non trivial.

Next we state four propositions:

(1) For every non empty set A and for every trivial non empty set B such that $A \subseteq B$ holds $A = B$.

(2) For every trivial non empty set A and for every set B such that $A \cap B$ is non empty holds $A \subseteq B$.

(3) For every 1-sorted structure Y holds Y is trivial iff the carrier of Y is trivial.

(4) Let Y_0, Y_1 be 1-sorted structures. Suppose the carrier of Y_0 = the carrier of Y_1. If Y_0 is trivial, then Y_1 is trivial.

Let S be a set. An element of S is proper if:

(Def.2) It $\neq \bigcup S$.

Let S be a set. Observe that there exists a subset of S which is non proper.

Next we state the proposition

(5) For every set S and for every subset A of S holds A is proper iff $A \neq S$.

Let S be a non empty set. Observe that every subset of S which is non proper is also non empty and every subset of S which is empty is also proper.

Let S be a trivial non empty set. Observe that every subset of S which is proper is also empty and every subset of S which is non empty is also non proper.

Let S be a non empty set. One can check that there exists a subset of S which is proper and there exists a subset of S which is non proper.

Let S be a non empty set and let y be an element of S. Then $\{y\}$ is a non empty subset of S.

Let S be a non empty set. Observe that there exists a non empty subset of S which is trivial.

Let S be a non empty set and let y be an element of S. Then $\{y\}$ is a trivial non empty subset of S.

We now state two propositions:

(6) For every non empty set S and for every element y of S such that $\{y\}$ is proper holds S is non trivial.

(7) For every non trivial non empty set S and for every element y of S holds $\{y\}$ is proper.

Let S be a trivial non empty set. Note that every non empty subset of S is non proper and every non empty subset of S which is non proper is also trivial.

Let S be a non trivial non empty set. Observe that every non empty subset of S which is trivial is also proper and every non empty subset of S which is non proper is also non trivial.

Let S be a non trivial non empty set. One can check that there exists a non empty subset of S which is trivial and proper and there exists a non empty subset of S which is non trivial and non proper.

One can prove the following propositions:

(8) Let Y be a 1-sorted structure and let y be an element of the carrier of Y. If $\{y\}$ is proper, then Y is non trivial.

(9) For every non trivial 1-sorted structure Y and for every element y of the carrier of Y holds $\{y\}$ is proper.

Let Y be a trivial 1-sorted structure. Note that every non empty subset of Y is non proper and every non empty subset of Y which is non proper is also trivial.

Let Y be a non trivial 1-sorted structure. One can verify that every non empty subset of Y which is trivial is also proper and every non empty subset of Y which is non proper is also non trivial.

Let Y be a non trivial 1-sorted structure. One can check that there exists a non empty subset of Y which is trivial and proper and there exists a non empty subset of Y which is non trivial and non proper.

2. Proper Subspaces of Topological Spaces

The following three propositions are true:

(10) Let X be a topological structure and let X_0 be a subspace of X. Then the topological structure of X_0 is a strict subspace of X.

(11) Let X be a topological structure and let X_1, X_2 be subspaces of X. Suppose the carrier of X_1 = the carrier of X_2. Then the topological structure of X_1 = the topological structure of X_2.

(12) Let Y_0, Y_1 be topological structures. Suppose the topological structure of Y_0 = the topological structure of Y_1. If Y_0 is topological space-like, then Y_1 is topological space-like.

Let Y be a topological structure. A subspace of Y is proper if:

(Def.3) For every subset A of Y such that A = the carrier of it holds A is proper.

We now state three propositions:

(13) Let Y_0 be a subspace of Y and let A be a subset of Y. If A = the carrier of Y_0, then A is proper iff Y_0 is proper.

(14) Let Y_0, Y_1 be subspaces of Y. Suppose the topological structure of Y_0 = the topological structure of Y_1. If Y_0 is proper, then Y_1 is proper.

(15) For every subspace Y_0 of Y such that the carrier of Y_0 = the carrier of Y holds Y_0 is non proper.

Let Y be a trivial topological structure. Observe that every subspace of Y is non proper and every subspace of Y which is non proper is also trivial.

Let Y be a non trivial topological structure. Observe that every subspace of Y which is trivial is also proper and every subspace of Y which is non proper is also non trivial.

Let Y be a topological structure. Observe that there exists a subspace of Y which is non proper and strict.

Next we state the proposition

(16) For every non proper subspace Y_0 of Y holds the topological structure of Y_0 = the topological structure of Y.

Let Y be a topological structure. One can check the following observations:

* every subspace of Y which is discrete is also topological space-like,
* every subspace of Y which is anti-discrete is also topological space-like,
* every subspace of Y which is non topological space-like is also non discrete, and
* every subspace of Y which is non topological space-like is also non anti-discrete.

One can prove the following propositions:

(17) Let Y_0, Y_1 be topological structures. Suppose the topological structure of Y_0 = the topological structure of Y_1. If Y_0 is discrete, then Y_1 is discrete.

(18) Let Y_0, Y_1 be topological structures. Suppose the topological structure of Y_0 = the topological structure of Y_1. If Y_0 is anti-discrete, then Y_1 is anti-discrete.

Let Y be a topological structure. One can verify the following observations:

* every subspace of Y which is discrete is also almost discrete,
* every subspace of Y which is non almost discrete is also non discrete,
* every subspace of Y which is anti-discrete is also almost discrete, and
* every subspace of Y which is non almost discrete is also non anti-discrete.

One can prove the following proposition

(19) Let Y_0, Y_1 be topological structures. Suppose the topological structure of Y_0 = the topological structure of Y_1. If Y_0 is almost discrete, then Y_1 is almost discrete.

Let Y be a topological structure. One can check the following observations:

* every subspace of Y which is discrete and anti-discrete is also trivial,
* every subspace of Y which is anti-discrete and non trivial is also non discrete, and
* every subspace of Y which is discrete and non trivial is also non anti-discrete.

Let Y be a topological structure and let y be a point of Y. The functor Sspace(y) yielding a strict subspace of Y is defined as follows:

(Def.4) The carrier of Sspace(y) = $\{y\}$.

Let Y be a topological structure. Observe that there exists a subspace of Y which is trivial and strict.

Let Y be a topological structure and let y be a point of Y. Then Sspace(y) is a trivial strict subspace of Y.

We now state three propositions:

(20) For every topological structure Y and for every point y of Y holds Sspace(y) is proper iff $\{y\}$ is proper.

(21) For every topological structure Y and for every point y of Y such that Sspace(y) is proper holds Y is non trivial.

(22) For every non trivial topological structure Y and for every point y of Y holds Sspace(y) is proper.

Let Y be a non trivial topological structure. One can verify that there exists a subspace of Y which is proper trivial and strict.

We now state two propositions:

(23) Let Y be a topological structure and let Y_0 be a trivial subspace of Y. Suppose Y_0 is topological space-like. Then there exists a point y of Y such that the topological structure of Y_0 = the topological structure of Sspace(y).

(24) Let Y be a topological structure and let y be a point of Y. If Sspace(y) is topological space-like, then Sspace(y) is discrete and anti-discrete.

Let Y be a topological structure. Note that every subspace of Y which is trivial and topological space-like is also discrete and anti-discrete.

Let X be a topological space. Note that there exists a subspace of X which is trivial strict and topological space-like.

Let X be a topological space and let x be a point of X. Then Sspace(x) is a trivial strict topological space-like subspace of X.

Let X be a topological space. Observe that there exists a subspace of X which is discrete anti-discrete and strict.

Let X be a topological space and let x be a point of X. Then Sspace(x) is a discrete anti-discrete strict subspace of X.

Let X be a topological space. One can check the following observations:

* every subspace of X which is non proper is also open and closed,
* every subspace of X which is non open is also proper, and
* every subspace of X which is non closed is also proper.

Let X be a topological space. Note that there exists a subspace of X which is open closed and strict.

Let X be a discrete topological space. Note that every subspace of X which is anti-discrete is also trivial and every subspace of X which is non trivial is also non anti-discrete.

Let X be a discrete non trivial topological space. Observe that there exists a subspace of X which is discrete open closed proper and strict.

Let X be an anti-discrete topological space. One can check that every subspace of X which is discrete is also trivial and every subspace of X which is non trivial is also non discrete.

Let X be an anti-discrete non trivial topological space. One can verify that every proper subspace of X is non open and non closed and every discrete subspace of X is trivial and proper.

Let X be an anti-discrete non trivial topological space. One can check that there exists a subspace of X which is anti-discrete non open non closed proper and strict.

Let X be an almost discrete non trivial topological space. Observe that there exists a subspace of X which is almost discrete proper and strict.

3. Maximal Discrete Subsets and Subspaces

Let Y be a topological structure. A subset of Y is discrete if:

(Def.5) For every subset D of Y such that $D \subseteq$ it there exists a subset G of Y such that G is open and it $\cap G = D$.

Let Y be a topological structure. Let us observe that a subset of Y is discrete if:

(Def.6) For every subset D of Y such that $D \subseteq$ it there exists a subset F of Y such that F is closed and it $\cap F = D$.

We now state three propositions:

(25) Let Y_0, Y_1 be topological structures, and let D_0 be a subset of Y_0, and let D_1 be a subset of Y_1. Suppose the topological structure of $Y_0 =$ the topological structure of Y_1 and $D_0 = D_1$. If D_0 is discrete, then D_1 is discrete.

(26) Let Y be a topological structure, and let Y_0 be a subspace of Y, and let A be a subset of Y. Suppose $A =$ the carrier of Y_0. Then A is discrete if and only if Y_0 is discrete.

(27) Let Y be a topological structure and let A be a subset of Y. Suppose $A =$ the carrier of Y. Then A is discrete if and only if Y is discrete.

In the sequel Y will denote a topological structure.

We now state several propositions:

(28) For all subsets A, B of Y such that $B \subseteq A$ holds if A is discrete, then B is discrete.

(29) For all subsets A, B of Y such that A is discrete or B is discrete holds $A \cap B$ is discrete.

(30) Suppose that for all subsets P, Q of Y such that P is open and Q is open holds $P \cap Q$ is open and $P \cup Q$ is open. Let A, B be subsets of Y. Suppose A is open and B is open. If A is discrete and B is discrete, then $A \cup B$ is discrete.

(31) Suppose that for all subsets P, Q of Y such that P is closed and Q is closed holds $P \cap Q$ is closed and $P \cup Q$ is closed. Let A, B be subsets of Y. Suppose A is closed and B is closed. If A is discrete and B is discrete, then $A \cup B$ is discrete.

(32) Let A be a subset of Y. Suppose A is discrete. Let x be a point of Y. If $x \in A$, then there exists a subset G of Y such that G is open and $A \cap G = \{x\}$.

(33) Let A be a subset of Y. Suppose A is discrete. Let x be a point of Y. If $x \in A$, then there exists a subset F of Y such that F is closed and $A \cap F = \{x\}$.

In the sequel X denotes a topological space.

The following propositions are true:

(34) Let A_0 be a non empty subset of X. Suppose A_0 is discrete. Then there exists a discrete strict subspace X_0 of X such that A_0 = the carrier of X_0.

(35) Every empty subset of X is discrete.

(36) For every point x of X holds $\{x\}$ is discrete.

(37) Let A be a subset of X. Suppose that for every point x of X such that $x \in A$ there exists a subset G of X such that G is open and $A \cap G = \{x\}$. Then A is discrete.

(38) Let A, B be subsets of X. Suppose A is open and B is open. If A is discrete and B is discrete, then $A \cup B$ is discrete.

(39) Let A, B be subsets of X. Suppose A is closed and B is closed. If A is discrete and B is discrete, then $A \cup B$ is discrete.

(40) For every subset A of X such that A is everywhere dense holds if A is discrete, then A is open.

(41) For every subset A of X holds A is discrete iff for every subset D of X such that $D \subseteq A$ holds $A \cap \overline{D} = D$.

(42) For every subset A of X such that A is discrete and for every point x of X such that $x \in A$ holds $A \cap \overline{\{x\}} = \{x\}$.

(43) For every discrete topological space X holds every subset of X is discrete.

(44) Let X be an anti-discrete topological space and let A be a non empty subset of X. Then A is discrete if and only if A is trivial.

Let Y be a topological structure. A subset of Y is maximal discrete if:

(Def.7) It is discrete and for every subset D of Y such that D is discrete and it $\subseteq D$ holds it $= D$.

The following proposition is true

(45) Let Y_0, Y_1 be topological structures, and let D_0 be a subset of Y_0, and let D_1 be a subset of Y_1. Suppose the topological structure of Y_0 = the topological structure of Y_1 and $D_0 = D_1$. If D_0 is maximal discrete, then D_1 is maximal discrete.

In the sequel X will denote a topological space.

Next we state several propositions:

(46) Every empty subset of X is not maximal discrete.

(47) For every subset A of X such that A is open holds if A is maximal discrete, then A is dense.

(48) For every subset A of X such that A is dense holds if A is discrete, then A is maximal discrete.

(49) Let X be a discrete topological space and let A be a subset of X. Then A is maximal discrete if and only if A is non proper.

(50) Let X be an anti-discrete topological space and let A be a non empty subset of X. Then A is maximal discrete if and only if A is trivial.

Let Y be a topological structure. A subspace of Y is maximal discrete if:

(Def.8) For every subset A of Y such that $A =$ the carrier of it holds A is maximal discrete.

One can prove the following proposition

(51) Let Y be a topological structure, and let Y_0 be a subspace of Y, and let A be a subset of Y. Suppose $A =$ the carrier of Y_0. Then A is maximal discrete if and only if Y_0 is maximal discrete.

Let Y be a topological structure. Note that every subspace of Y which is maximal discrete is also discrete and every subspace of Y which is non discrete is also non maximal discrete.

Next we state two propositions:

(52) Let X_0 be a subspace of X. Then X_0 is maximal discrete if and only if the following conditions are satisfied:

(i) X_0 is discrete, and

(ii) for every discrete subspace Y_0 of X such that X_0 is a subspace of Y_0 holds the topological structure of $X_0 =$ the topological structure of Y_0.

(53) Let A_0 be a non empty subset of X. Suppose A_0 is maximal discrete. Then there exists a strict subspace X_0 of X such that X_0 is maximal discrete and $A_0 =$ the carrier of X_0.

Let X be a discrete topological space. One can verify the following observations:

* every subspace of X which is maximal discrete is also non proper,
* every subspace of X which is proper is also non maximal discrete,
* every subspace of X which is non proper is also maximal discrete, and
* every subspace of X which is non maximal discrete is also proper.

Let X be an anti-discrete topological space. One can check the following observations:

* every subspace of X which is maximal discrete is also trivial,
* every subspace of X which is non trivial is also non maximal discrete,
* every subspace of X which is trivial is also maximal discrete, and

* every subspace of X which is non maximal discrete is also non trivial.

4. Maximal Discrete Subspaces of Almost Discrete Spaces

The scheme *ExChoiceFCol* deals with a topological structure $\mathcal{A}$, a family $\mathcal{B}$ of subsets of $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

There exists a function f from $\mathcal{B}$ into the carrier of $\mathcal{A}$ such that for every subset S of $\mathcal{A}$ such that $S \in \mathcal{B}$ holds $\mathcal{P}[S, f(S)]$

provided the following condition is met:

- For every subset S of $\mathcal{A}$ such that $S \in \mathcal{B}$ there exists a point x of $\mathcal{A}$ such that $\mathcal{P}[S, x]$.

In the sequel X will denote an almost discrete topological space.

We now state a number of propositions:

(54) For every subset A of X holds $\overline{A} = \bigcup\{\overline{\{a\}} : a$ ranges over points of X, $a \in A\}$.

(55) For all points a, b of X such that $a \in \overline{\{b\}}$ holds $\overline{\{a\}} = \overline{\{b\}}$.

(56) For all points a, b of X holds $\overline{\{a\}} \cap \overline{\{b\}} = \emptyset$ or $\overline{\{a\}} = \overline{\{b\}}$.

(57) Let A be a subset of X. Suppose that for every point x of X such that $x \in A$ there exists a subset F of X such that F is closed and $A \cap F = \{x\}$. Then A is discrete.

(58) For every subset A of X such that for every point x of X such that $x \in A$ holds $A \cap \overline{\{x\}} = \{x\}$ holds A is discrete.

(59) Let A be a subset of X. Then A is discrete if and only if for all points a, b of X such that $a \in A$ and $b \in A$ holds if $a \neq b$, then $\overline{\{a\}} \cap \overline{\{b\}} = \emptyset$.

(60) Let A be a subset of X. Then A is discrete if and only if for every point x of X such that $x \in \overline{A}$ there exists a point a of X such that $a \in A$ and $A \cap \overline{\{x\}} = \{a\}$.

(61) For every subset A of X such that A is open or closed holds if A is maximal discrete, then A is not proper.

(62) For every subset A of X such that A is maximal discrete holds A is dense.

(63) For every subset A of X such that A is maximal discrete holds $\bigcup\{\overline{\{a\}} : a$ ranges over points of X, $a \in A\} =$ the carrier of X.

(64) Let A be a subset of X. Then A is maximal discrete if and only if for every point x of X there exists a point a of X such that $a \in A$ and $A \cap \overline{\{x\}} = \{a\}$.

(65) For every subset A of X such that A is discrete there exists a subset M of X such that $A \subseteq M$ and M is maximal discrete.

(66) There exists subset of X which is maximal discrete.

(67) Let Y_0 be a discrete subspace of X. Then there exists a strict subspace X_0 of X such that Y_0 is a subspace of X_0 and X_0 is maximal discrete.

Let X be an almost discrete non discrete topological space. One can verify that every subspace of X which is maximal discrete is also proper and every subspace of X which is non proper is also non maximal discrete.

Let X be an almost discrete non anti-discrete topological space. Observe that every subspace of X which is maximal discrete is also non trivial and every subspace of X which is trivial is also non maximal discrete.

Let X be an almost discrete topological space. Note that there exists a subspace of X which is maximal discrete and strict.

5. Continuous Mappings and Almost Discrete Spaces

The scheme *MapExChoiceF* concerns a topological structure $\mathcal{A}$, a topological structure $\mathcal{B}$, and a binary predicate $\mathcal{P}$, and states that:

> There exists a map f from $\mathcal{A}$ into $\mathcal{B}$ such that for every point x of $\mathcal{A}$ holds $\mathcal{P}[x, f(x)]$

provided the parameters have the following property:

- For every point x of $\mathcal{A}$ there exists a point y of $\mathcal{B}$ such that $\mathcal{P}[x, y]$.

In the sequel X, Y are topological spaces.

Next we state four propositions:

(68) For every discrete topological space X holds every mapping from X into Y is continuous.

(69) If for every topological space Y holds every mapping from X into Y is continuous, then X is discrete.

(70) For every anti-discrete topological space Y holds every mapping from X into Y is continuous.

(71) If for every topological space X holds every mapping from X into Y is continuous, then Y is anti-discrete.

In the sequel X will be a discrete topological space and X_0 will be a subspace of X.

One can prove the following two propositions:

(72) There exists continuous mapping from X into X_0 which is a retraction.

(73) X_0 is a retract of X.

In the sequel X will be an almost discrete topological space and X_0 will be a maximal discrete subspace of X.

Next we state four propositions:

(74) There exists continuous mapping from X into X_0 which is a retraction.

(75) X_0 is a retract of X.

(76) Let r be a continuous mapping from X into X_0. Suppose r is a retraction. Let F be a subset of X_0 and let E be a subset of X. If $F = E$, then $r^{-1} F = \overline{E}$.

(77) Let r be a continuous mapping from X into X_0. Suppose r is a retraction. Let a be a point of X_0 and let b be a point of X. If $a = b$, then $r^{-1}\{a\} = \overline{\{b\}}$.

In the sequel X_0 is a discrete subspace of X.

The following two propositions are true:

(78) There exists continuous mapping from X into X_0 which is a retraction.

(79) X_0 is a retract of X.

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