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Product of Family of Universal Algebras

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Summary. The product of two algebras, trivial algebra determined by an empty set and product of a family of algebras are defined. Some basic properties are shown.

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The terminology and notation used in this paper have been introduced in the following articles: [14], [6], [3], [7], [11], [15], [12], [9], [5], [8], [1], [2], [10], [4], and [13].

1. Product of Two Algebras

The following proposition is true

(1) For all non-empty set D_1, D_2 and for all natural numbers n, m such that ${D_1}^n = {D_2}^m$ holds $n = m$.

For simplicity we follow a convention: U_1, U_2, U_3 denote universal algebras, k, m, i denote natural numbers, z is arbitrary, and h_1, h_2 denote finite sequences of elements of $[\!: A, B :\!]$.

Let us consider A, B and let us consider h_1. The functor $\pi_1(h_1)$ yielding a finite sequence of elements of A is defined as follows:

(Def.1) $\mathrm{len}\,\pi_1(h_1) = \mathrm{len}\,h_1$ and for every n such that $n \in \mathrm{dom}\,\pi_1(h_1)$ holds $(\pi_1(h_1))(n) = h_1(n)_{\mathbf{1}}$.

The functor $\pi_2(h_1)$ yielding a finite sequence of elements of B is defined as follows:

(Def.2) $\mathrm{len}\,\pi_2(h_1) = \mathrm{len}\,h_1$ and for every n such that $n \in \mathrm{dom}\,\pi_2(h_1)$ holds $(\pi_2(h_1))(n) = h_1(n)_{\mathbf{2}}$.

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Let us consider A, B, let f_1 be a homogeneous quasi total non-empty partial function from A^* to A, and let f_2 be a homogeneous quasi total non-empty partial function from B^* to B. Let us assume that $\operatorname{arity} f_1 = \operatorname{arity} f_2$. The functor $]\!] f_1, f_2 [\![$ yielding a homogeneous quasi total non-empty partial function from $[\!: A, B :\!]^*$ to $[\!: A, B :\!]$ is defined by the conditions (Def.3).

(Def.3) (i) $\operatorname{dom}]\!] f_1, f_2 [\![= [\!: A, B :\!]^{\operatorname{arity} f_1}$, and

(ii) for every finite sequence h of elements of $[\!: A, B :\!]$ such that $h \in \operatorname{dom}]\!] f_1, f_2 [\![$ holds $]\!] f_1, f_2 [\![(h) = \langle f_1(\pi_1(h)), f_2(\pi_2(h)) \rangle$.

In the sequel h_1 will denote a homogeneous quasi total non-empty partial function from (the carrier of U_1)* to the carrier of U_1.

Let us consider U_1, U_2. Let us assume that U_1 and U_2 are similar. The functor $\operatorname{Opers}(U_1, U_2)$ yielding a finite sequence of elements of $[\!:$ the carrier of U_1, the carrier of $U_2 :\!]^* \dot{\rightarrow} [\!:$ the carrier of U_1, the carrier of $U_2 :\!]$ is defined as follows:

(Def.4) $\operatorname{len} \operatorname{Opers}(U_1, U_2) = \operatorname{len} \operatorname{Opers} U_1$ and for every n such that $n \in \operatorname{dom} \operatorname{Opers}(U_1, U_2)$ and for all h_1, h_2 such that $h_1 = (\operatorname{Opers} U_1)(n)$ and $h_2 = (\operatorname{Opers} U_2)(n)$ holds $(\operatorname{Opers}(U_1, U_2))(n) =]\!] h_1, h_2 [\![$.

The following proposition is true

(2) If U_1 and U_2 are similar, then $\langle [\!:$ the carrier of U_1, the carrier of $U_2 :\!]$, $\operatorname{Opers}(U_1, U_2) \rangle$ is a strict universal algebra.

Let us consider U_1, U_2. Let us assume that U_1 and U_2 are similar. The functor $[\!: U_1, U_2 :\!]$ yielding a strict universal algebra is defined as follows:

(Def.5) $[\!: U_1, U_2 :\!] = \langle [\!:$ the carrier of U_1, the carrier of $U_2 :\!], \operatorname{Opers}(U_1, U_2) \rangle$.

Let A, B be non-empty set. The functor $\operatorname{Inv}(A, B)$ yielding a function from $[\!: A, B :\!]$ into $[\!: B, A :\!]$ is defined as follows:

(Def.6) For every element a of $[\!: A, B :\!]$ holds $(\operatorname{Inv}(A, B))(a) = \langle a_2, a_1 \rangle$.

One can prove the following propositions:

(3) For all non-empty set A, B holds $\operatorname{rng} \operatorname{Inv}(A, B) = [\!: B, A :\!]$.

(4) For all non-empty set A, B holds $\operatorname{Inv}(A, B)$ is one-to-one.

(5) Suppose U_1 and U_2 are similar. Then Inv(the carrier of U_1, the carrier of U_2) is a function from the carrier of $[\!: U_1, U_2 :\!]$ into the carrier of $[\!: U_2, U_1 :\!]$.

(6) Suppose U_1 and U_2 are similar. Let o_1 be a operation of U_1, and let o_2 be a operation of U_2, and let o be a operation of $[\!: U_1, U_2 :\!]$, and let n be a natural number. Suppose $o_1 = (\operatorname{Opers} U_1)(n)$ and $o_2 = (\operatorname{Opers} U_2)(n)$ and $o = (\operatorname{Opers} [\!: U_1, U_2 :\!])(n)$ and $n \in \operatorname{dom} \operatorname{Opers} U_1$. Then $\operatorname{arity} o = \operatorname{arity} o_1$ and $\operatorname{arity} o = \operatorname{arity} o_2$ and $o =]\!] o_1, o_2 [\![$.

(7) If U_1 and U_2 are similar, then $[\!: U_1, U_2 :\!]$ and U_1 are similar.

(8) Let U_1, U_2, U_3, U_4 be universal algebras. Suppose U_1 is a subalgebra of U_2 and U_3 is a subalgebra of U_4 and U_2 and U_4 are similar. Then $[\!: U_1, U_3 :\!]$ is a subalgebra of $[\!: U_2, U_4 :\!]$.

2. Trivial Algebra

Let k be a natural number. The functor $\mathrm{TrivOp}(k)$ yields a homogeneous quasi total non-empty partial function from $\{\emptyset\}^*$ to $\{\emptyset\}$ and is defined as follows:

(Def.7) $\mathrm{dom}\,\mathrm{TrivOp}(k) = \{k \longmapsto \emptyset\}$ and $\mathrm{rng}\,\mathrm{TrivOp}(k) = \{\emptyset\}$.

The following proposition is true

(9) $\mathrm{arity}\,\mathrm{TrivOp}(k) = k$.

Let f be a finite sequence of elements of $\mathbb{N}$. The functor $\mathrm{TrivOps}(f)$ yielding a finite sequence of elements of $\{\emptyset\}^* \dot{\rightarrow} \{\emptyset\}$ is defined as follows:

(Def.8) $\mathrm{len}\,\mathrm{TrivOps}(f) = \mathrm{len}\,f$ and for every n such that $n \in \mathrm{dom}\,\mathrm{TrivOps}(f)$ and for every m such that $m = f(n)$ holds $(\mathrm{TrivOps}(f))(n) = \mathrm{TrivOp}(m)$.

We now state two propositions:

(10) For every finite sequence f of elements of $\mathbb{N}$ holds $\mathrm{TrivOps}(f)$ is homogeneous quasi total and non-empty.

(11) For every finite sequence f of elements of $\mathbb{N}$ such that $f \neq \varepsilon$ holds $\langle\{\emptyset\}, \mathrm{TrivOps}(f)\rangle$ is a strict universal algebra.

Let D be a non empty set. Observe that there exists a finite sequence of elements of D which is non empty and there exists an element of D^* which is non empty.

Let f be a non empty finite sequence of elements of $\mathbb{N}$. The trivial algebra of f yielding a strict universal algebra is defined as follows:

(Def.9) The trivial algebra of $f = \langle\{\emptyset\}, \mathrm{TrivOps}(f)\rangle$.

3. Product of Universal Algebras

A function is universal algebra yielding if:

(Def.10) For every x such that $x \in \mathrm{dom}\,\mathrm{it}$ holds $\mathrm{it}(x)$ is a universal algebra.

A function is 1-sorted yielding if:

(Def.11) For every x such that $x \in \mathrm{dom}\,\mathrm{it}$ holds $\mathrm{it}(x)$ is a 1-sorted structure.

One can check that there exists a function which is universal algebra yielding.

One can verify that every function which is universal algebra yielding is also 1-sorted yielding.

Let I be a set. Observe that there exists a many sorted set of I which is 1-sorted yielding.

A function is equal signature if:

(Def.12) For all x, y such that $x \in \mathrm{dom}\,\mathrm{it}$ and $y \in \mathrm{dom}\,\mathrm{it}$ and for all U_1, U_2 such that $U_1 = \mathrm{it}(x)$ and $U_2 = \mathrm{it}(y)$ holds $\mathrm{signature}\,U_1 = \mathrm{signature}\,U_2$.

Let J be a non-empty set. One can check that there exists a many sorted set of J which is equal signature and universal algebra yielding.

Let J be a non empty set, let A be a universal algebra yielding many sorted set of J, and let j be an element of J. Then $A(j)$ is a universal algebra.

Let J be a non-empty set and let A be a universal algebra yielding many sorted set of J. The functor support A yields a non-empty many sorted set of J and is defined as follows:

(Def.13) For every element j of J holds $(\text{support}\, A)(j) =$ the carrier of $A(j)$.

Let J be a non-empty set and let A be an equal signature universal algebra yielding many sorted set of J. The functor $\text{ComSign}(A)$ yields a finite sequence of elements of $\mathbb{N}$ and is defined as follows:

(Def.14) For every element j of J holds $\text{ComSign}(A) = \text{signature}\, A(j)$.

A function is function yielding if:

(Def.15) For every x such that $x \in \text{dom}\, \text{it}$ holds $\text{it}(x)$ is a function.

Let us note that there exists a function which is function yielding.

Let I be a set. Note that there exists a many sorted set of I which is function yielding.

Let I be a set. A many sorted function of I is a function yielding many sorted set of I.

Let J be a non-empty set, let B be a many sorted function of J, and let j be an element of J. Then $B(j)$ is a function.

Let J be a non-empty set, let B be a non-empty many sorted set of J, and let j be an element of J. Then $B(j)$ is a non-empty set.

Let J be a non-empty set and let B be a non-empty many sorted set of J. Then $\prod B$ is a non-empty set.

Let J be a non-empty set and let B be a non-empty many sorted set of J. A many sorted function of J is said to be a many sorted operation of B if:

(Def.16) For every element j of J holds $\text{it}(j)$ is a homogeneous quasi total non-empty partial function from $B(j)^*$ to $B(j)$.

Let J be a non-empty set, let B be a non-empty many sorted set of J, let O be a many sorted operation of B, and let j be an element of J. Then $O(j)$ is a homogeneous quasi total non-empty partial function from $B(j)^*$ to $B(j)$.

A function is equal arity if satisfies the condition (Def.17).

(Def.17) Let x, y be arbitrary. Suppose $x \in \text{dom}\, \text{it}$ and $y \in \text{dom}\, \text{it}$. Let f, g be functions. Suppose $\text{it}(x) = f$ and $\text{it}(y) = g$. Let n, m be natural numbers and let X, Y be non-empty set. Suppose $\text{dom}\, f = X^n$ and $\text{dom}\, g = Y^m$. Let o_1 be a homogeneous quasi total non-empty partial function from X^* to X and let o_2 be a homogeneous quasi total non-empty partial function from Y^* to Y. If $f = o_1$ and $g = o_2$, then $\text{arity}\, o_1 = \text{arity}\, o_2$.

Let J be a non-empty set and let B be a non-empty many sorted set of J. One can verify that there exists a many sorted operation of B which is equal arity.

The following proposition is true

(12) Let J be a non-empty set, and let B be a non-empty many sorted set of J, and let O be a many sorted operation of B. Then O is equal arity

if and only if for all elements i, j of J holds arity $O(i) =$ arity $O(j)$.

Let I be a set, let f be a many sorted function of I, and let x be a many sorted set of I. The functor $f \leftrightarrow\!\!\!\!\!\text{P}\ x$ yields a many sorted set of I and is defined as follows:

(Def.18) For arbitrary i such that $i \in I$ and for every function g such that $g = f(i)$ holds $(f \leftrightarrow\!\!\!\!\!\text{P}\ x)(i) = g(x(i))$.

Let J be a non-empty set, let B be a non-empty many sorted set of J, and let p be a finite sequence of elements of $\prod B$. Then uncurry p is a many sorted set of $[\!:\operatorname{dom} p, J:\!]$.

Let I, J be sets and let X be a many sorted set of $[\!:I, J:\!]$. Then $\curvearrowleft X$ is a many sorted set of $[\!:J, I:\!]$.

Let X be a set, let Y be a non-empty set, and let f be a many sorted set of $[\!:X, Y:\!]$. Then curry f is a many sorted set of X.

Let J be a non-empty set, let B be a non-empty many sorted set of J, and let O be an equal arity many sorted operation of B. The functor $\mathrm{ComAr}(O)$ yielding a natural number is defined as follows:

(Def.19) For every element j of J holds $\mathrm{ComAr}(O) =$ arity $O(j)$.

Let I be a set and let A be a many sorted set of I. The functor ε_A yielding a many sorted set of I is defined as follows:

(Def.20) For arbitrary i such that $i \in I$ holds $\varepsilon_A(i) = \varepsilon_{A(i)}$.

Let J be a non-empty set, let B be a non-empty many sorted set of J, and let O be an equal arity many sorted operation of B. The functor $]\!]O[\![$ yielding a homogeneous quasi total non-empty partial function from $(\prod B)^*$ to $\prod B$ is defined by the conditions (Def.21).

(Def.21) (i) $\operatorname{dom}]\!]O[\![= (\prod B)^{\mathrm{ComAr}(O)}$, and

(ii) for every element p of $(\prod B)^*$ such that $p \in \operatorname{dom}]\!]O[\![$ holds if $\operatorname{dom} p = \emptyset$, then $]\!]O[\![(p) = O \leftrightarrow\!\!\!\!\!\text{P}\ (\varepsilon_B)$ and if $\operatorname{dom} p \neq \emptyset$, then for every non-empty set Z and for every many sorted set w of $[\!:J, Z:\!]$ such that $Z = \operatorname{dom} p$ and $w = \curvearrowleft$uncurry p holds $]\!]O[\![(p) = O \leftrightarrow\!\!\!\!\!\text{P}$ curry w.

Let J be a non-empty set, let A be an equal signature universal algebra yielding many sorted set of J, and let n be a natural number. Let us assume that $n \in$ Seg len ComSign(A). The functor ProdOp(A, n) yielding an equal arity many sorted operation of support A is defined by:

(Def.22) For every element j of J and for every operation o of $A(j)$ such that $(\mathrm{Opers}\, A(j))(n) = o$ holds $(\mathrm{ProdOp}(A, n))(j) = o$.

Let J be a non-empty set and let A be an equal signature universal algebra yielding many sorted set of J. The functor ProdOpSeq(A) yielding a finite sequence of elements of $(\prod \text{support } A)^* \dot\rightarrow \prod \text{support } A$ is defined as follows:

(Def.23) len ProdOpSeq(A) = len ComSign(A) and for every n such that $n \in$ dom ProdOpSeq(A) holds $(\mathrm{ProdOpSeq}(A))(n) =]\!]\mathrm{ProdOp}(A, n)[\![$.

Let J be a non-empty set and let A be an equal signature universal algebra yielding many sorted set of J. The functor ProdUnivAlg(A) yields a strict universal algebra and is defined as follows:

(Def.24) $\mathrm{ProdUnivAlg}(A) = \langle \prod \mathrm{support}\, A, \mathrm{ProdOpSeq}(A) \rangle$.

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