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Free Universal Algebra Construction

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Summary. A construction of the free universal algebra with fixed signature and a given set of generators.

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The articles [17], [19], [20], [9], [13], [10], [11], [5], [16], [8], [18], [1], [3], [4], [2], [15], [7], [12], [6], and [14] provide the terminology and notation for this paper.

1. Preliminaries

In the sequel x is arbitrary and n denotes a natural number.

Let D be a non empty set and let X be a set. Then $D \cup X$ is a non empty set.

A set is missing $\mathbb{N}$ if:

(Def.1) It $\cap \mathbb{N} = \emptyset$.

One can check that there exists a set which is non empty and missing $\mathbb{N}$.

A finite sequence has zero if:

(Def.2) $0 \in \mathrm{rng}\,\mathrm{it}$.

Let us observe that there exists a finite sequence of elements of $\mathbb{N}$ which is non empty and has zero and there exists a finite sequence of elements of $\mathbb{N}$ which is non empty and without zero.

Let f be a non empty finite sequence. Then $\mathrm{dom}\, f$ is a non empty set.

Let X be a set, let D be a non empty set, let f be a partial function from X to D, and let x be arbitrary. Let us assume that $x \in \mathrm{dom}\, f$. The functor $\pi_x f$ yields an element of D and is defined as follows:

(Def.3) $\pi_x f = f(x)$.

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2. Free Universal Algebra - General Notions

Let U_1 be a universal algebra and let n be a natural number. Let us assume that $n \in \mathrm{dom\,Opers}\, U_1$. The functor $\mathrm{oper}(n, U_1)$ yielding a operation of U_1 is defined as follows:

(Def.4) $\mathrm{oper}(n, U_1) = (\mathrm{Opers}\, U_1)(n)$.

Let U_0 be a universal algebra. A subset of U_0 is called a generator set of U_0 if:

(Def.5) The carrier of $\mathrm{Gen}^{\mathrm{UA}}(\mathrm{it})$ = the carrier of U_0.

Let U_0 be a universal algebra. A generator set of U_0 is free if satisfies the condition (Def.6).

(Def.6) Let U_1 be a universal algebra. Suppose U_0 and U_1 are similar. Let f be a function from it into the carrier of U_1. Then there exists a function h from U_0 into U_1 such that h is a homomorphism of U_0 into U_1 and $h \restriction \mathrm{it} = f$.

A universal algebra is free if:

(Def.7) There exists generator set of it which is free.

Let us observe that there exists a universal algebra which is free and strict.

Let U_0 be a free universal algebra. Observe that there exists a generator set of U_0 which is free.

One can prove the following proposition

(1) Let U_0 be a strict universal algebra and let A be a subset of U_0. Then A is a generator set of U_0 if and only if $\mathrm{Gen}^{\mathrm{UA}}(A) = U_0$.

3. Construction of Decorated Tree Structure for Free Universal Algebra

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let X be a set. The functor $\mathrm{REL}(f, X)$ yielding a relation between $\mathrm{dom}\, f \cup X$ and $(\mathrm{dom}\, f \cup X)^*$ is defined by:

(Def.8) For every element a of $\mathrm{dom}\, f \cup X$ and for every element b of $(\mathrm{dom}\, f \cup X)^*$ holds $\langle a, b \rangle \in \mathrm{REL}(f, X)$ iff $a \in \mathrm{dom}\, f$ and $f(a) = \mathrm{len}\, b$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let X be a set. The functor $\mathrm{DTConUA}(f, X)$ yields a strict tree construction structure and is defined as follows:

(Def.9) $\mathrm{DTConUA}(f, X) = \langle \mathrm{dom}\, f \cup X, \mathrm{REL}(f, X) \rangle$.

Next we state two propositions:

(2) Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let X be a set. Then the terminals of $\mathrm{DTConUA}(f, X) \subseteq X$ and the nonterminals of $\mathrm{DTConUA}(f, X) = \mathrm{dom}\, f$.

(3) Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let X be a missing $\mathbb{N}$ set. Then the terminals of $\mathrm{DTConUA}(f, X) = X$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let X be a set. Then $\mathrm{DTConUA}(f, X)$ is a strict tree construction structure with nonterminals.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let X be a set. Then $\mathrm{DTConUA}(f, X)$ is a strict tree construction structure with nonterminals and useful nonterminals.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a missing $\mathbb{N}$ non empty set. Then $\mathrm{DTConUA}(f, D)$ is a strict tree construction structure with terminals, nonterminals, and useful nonterminals.

Let f be a non empty finite sequence of elements of $\mathbb{N}$, let X be a set, and let n be a natural number. Let us assume that $n \in \mathrm{dom}\, f$. The functor $\mathrm{Sym}(n, f, X)$ yielding a symbol of $\mathrm{DTConUA}(f, X)$ is defined by:

(Def.10) $\mathrm{Sym}(n, f, X) = n$.

4. Construction of Free Universal Algebra for Non-empty Set of Generators and Given Signature

Let f be a non empty finite sequence of elements of $\mathbb{N}$, let D be a missing $\mathbb{N}$ non empty set, and let n be a natural number. Let us assume that $n \in \mathrm{dom}\, f$. The functor $\mathrm{FreeOpNSG}(n, f, D)$ yields a homogeneous quasi total non empty partial function from $\mathrm{TS}(\mathrm{DTConUA}(f, D))^*$ to $\mathrm{TS}(\mathrm{DTConUA}(f, D))$ and is defined by the conditions (Def.11).

(Def.11) (i) $\mathrm{dom}\,\mathrm{FreeOpNSG}(n, f, D) = \mathrm{TS}(\mathrm{DTConUA}(f, D))^{\pi_n f}$, and

(ii) for every finite sequence p of elements of $\mathrm{TS}(\mathrm{DTConUA}(f, D))$ such that $p \in \mathrm{dom}\,\mathrm{FreeOpNSG}(n, f, D)$ holds $(\mathrm{FreeOpNSG}(n, f, D))(p) = (\mathrm{Sym}(n, f, D))\text{-tree}(p)$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a missing $\mathbb{N}$ non empty set. The functor $\mathrm{FreeOpSeqNSG}(f, D)$ yielding a finite sequence of elements of $\mathrm{TS}(\mathrm{DTConUA}(f, D))^* \dot{\rightarrow} \mathrm{TS}(\mathrm{DTConUA}(f, D))$ is defined as follows:

(Def.12) $\mathrm{len}\,\mathrm{FreeOpSeqNSG}(f, D) = \mathrm{len}\, f$ and for every n such that $n \in \mathrm{dom}\,\mathrm{FreeOpSeqNSG}(f, D)$ holds $(\mathrm{FreeOpSeqNSG}(f, D))(n) = \mathrm{FreeOpNSG}(n, f, D)$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a missing $\mathbb{N}$ non empty set. The functor $\mathrm{FreeUnivAlgNSG}(f, D)$ yields a strict universal algebra and is defined as follows:

(Def.13) $\mathrm{FreeUnivAlgNSG}(f, D) = \langle \mathrm{TS}(\mathrm{DTConUA}(f, D)), \mathrm{FreeOpSeqNSG}(f, D) \rangle$.

One can prove the following proposition

(4) For every non empty finite sequence f of elements of $\mathbb{N}$ and for every missing $\mathbb{N}$ non empty set D holds signature $\mathrm{FreeUnivAlgNSG}(f, D) = f$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a non empty missing $\mathbb{N}$ set. The functor FreeGenSetNSG(f, D) yielding a subset of FreeUnivAlgNSG(f, D) is defined by:

(Def.14) FreeGenSetNSG$(f, D) = \{$the root tree of s: s ranges over symbols of DTConUA(f, D), $s \in$ the terminals of DTConUA$(f, D)\}$.

One can prove the following proposition

(5) Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a non empty missing $\mathbb{N}$ set. Then FreeGenSetNSG(f, D) is non empty.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a non empty missing $\mathbb{N}$ set. Then FreeGenSetNSG(f, D) is a generator set of FreeUnivAlgNSG(f, D).

Let f be a non empty finite sequence of elements of $\mathbb{N}$, let D be a non empty missing $\mathbb{N}$ set, let C be a non empty set, let s be a symbol of DTConUA(f, D), and let F be a function from FreeGenSetNSG(f, D) into C. Let us assume that $s \in$ the terminals of DTConUA(f, D). The functor $\pi_s F$ yielding an element of C is defined as follows:

(Def.15) $\pi_s F = F($the root tree of $s)$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$, let D be a non empty missing $\mathbb{N}$ set, and let s be a symbol of DTConUA(f, D). Let us assume that there exists a finite sequence p such that $s \Rightarrow p$. The functor $^{@}s$ yielding a natural number is defined by:

(Def.16) $^{@}s = s$.

Next we state the proposition

(6) For every non empty finite sequence f of elements of $\mathbb{N}$ and for every non empty missing $\mathbb{N}$ set D holds FreeGenSetNSG(f, D) is free.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a non empty missing $\mathbb{N}$ set. Then FreeUnivAlgNSG(f, D) is a strict free universal algebra.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ and let D be a non empty missing $\mathbb{N}$ set. Then FreeGenSetNSG(f, D) is a free generator set of FreeUnivAlgNSG(f, D).

5. Construction of Free Universal Algebra and Set of Generators

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero, let D be a missing $\mathbb{N}$ set, and let n be a natural number. Let us assume that $n \in \operatorname{dom} f$. The functor FreeOpZAO$(n, f, D)$ yields a homogeneous quasi total non empty partial function from TS(DTConUA$(f, D))^*$ to TS(DTConUA(f, D)) and is defined by the conditions (Def.17).

(Def.17) (i) dom FreeOpZAO$(n, f, D) = $ TS(DTConUA$(f, D))^{\pi_n f}$, and

(ii) for every finite sequence p of elements of TS(DTConUA(f, D)) such that $p \in$ dom FreeOpZAO(n, f, D) holds (FreeOpZAO(n, f, D))(p) = (Sym(n, f, D))-tree(p).

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. The functor FreeOpSeqZAO(f, D) yields a finite sequence of elements of TS(DTConUA(f, D))$^* \dot{\rightarrow}$ TS(DTConUA(f, D)) and is defined by:

(Def.18) len FreeOpSeqZAO(f, D) = len f and for every n such that $n \in$ dom FreeOpSeqZAO(f, D) holds (FreeOpSeqZAO(f, D))(n) = FreeOpZAO(n, f, D).

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. The functor FreeUnivAlgZAO(f, D) yielding a strict universal algebra is defined by:

(Def.19) FreeUnivAlgZAO(f, D) = $\langle$TS(DTConUA(f, D)), FreeOpSeqZAO(f, D)$\rangle$

We now state three propositions:

(7) For every non empty finite sequence f of elements of $\mathbb{N}$ with zero and for every missing $\mathbb{N}$ set D holds signature FreeUnivAlgZAO(f, D) = f.

(8) Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. Then FreeUnivAlgZAO(f, D) has constants.

(9) For every non empty finite sequence f of elements of $\mathbb{N}$ with zero and for every missing $\mathbb{N}$ set D holds Constants(FreeUnivAlgZAO(f, D)) $\neq \emptyset$.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. The functor FreeGenSetZAO(f, D) yielding a subset of FreeUnivAlgZAO(f, D) is defined as follows:

(Def.20) FreeGenSetZAO(f, D) = {the root tree of s: s ranges over symbols of DTConUA(f, D), $s \in$ the terminals of DTConUA(f, D)}.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. Then FreeGenSetZAO(f, D) is a generator set of FreeUnivAlgZAO(f, D).

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero, let D be a missing $\mathbb{N}$ set, let C be a non empty set, let s be a symbol of DTConUA(f, D), and let F be a function from FreeGenSetZAO(f, D) into C. Let us assume that $s \in$ the terminals of DTConUA(f, D). The functor $\pi_s F$ yields an element of C and is defined by:

(Def.21) $\pi_s F = F$(the root tree of s).

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero, let D be a missing $\mathbb{N}$ set, and let s be a symbol of DTConUA(f, D). Let us assume that there exists a finite sequence p such that $s \Rightarrow p$. The functor $^{@}s$ yields a natural number and is defined by:

(Def.22) $^{@}s = s$.

The following proposition is true

(10) For every non empty finite sequence f of elements of $\mathbb{N}$ with zero and for every missing $\mathbb{N}$ set D holds FreeGenSetZAO(f, D) is free.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. Then FreeUnivAlgZAO(f, D) is a strict free universal algebra.

Let f be a non empty finite sequence of elements of $\mathbb{N}$ with zero and let D be a missing $\mathbb{N}$ set. Then FreeGenSetZAO(f, D) is a free generator set of FreeUnivAlgZAO(f, D).

One can verify that there exists a universal algebra which is strict and free and has constants.

References

[1] Grzegorz Bancerek. Introduction to trees. *Formalized Mathematics*, 1(**2**):421–427, 1990.
[2] Grzegorz Bancerek. Joining of decorated trees. *Formalized Mathematics*, 4(**1**):77–82, 1993.
[3] Grzegorz Bancerek. König's lemma. *Formalized Mathematics*, 2(**3**):397–402, 1991.
[4] Grzegorz Bancerek. Sets and functions of trees and joining operations of trees. *Formalized Mathematics*, 3(**2**):195–204, 1992.
[5] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[6] Grzegorz Bancerek and Piotr Rudnicki. On defining functions on trees. *Formalized Mathematics*, 4(**1**):91–101, 1993.
[7] Ewa Burakowska. Subalgebras of the universal algebra. Lattices of subalgebras. *Formalized Mathematics*, 4(**1**):23–27, 1993.
[8] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[9] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[10] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[11] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[12] Patricia L. Carlson and Grzegorz Bancerek. Context-free grammar - part 1. *Formalized Mathematics*, 2(**5**):683–687, 1991.
[13] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[14] Małgorzata Korolkiewicz. Homomorphisms of algebras. Quotient universal algebra. *Formalized Mathematics*, 4(**1**):109–113, 1993.
[15] Jarosław Kotowicz, Beata Madras, and Małgorzata Korolkiewicz. Basic notation of universal algebra. *Formalized Mathematics*, 3(**2**):251–253, 1992.
[16] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[17] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[18] Wojciech A. Trybulec. Pigeon hole principle. *Formalized Mathematics*, 1(**3**):575–579, 1990.
[19] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[20] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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