FORMALIZED MATHEMATICS
Volume 3, Number 2, 1992
Université Catholique de Louvain

Product of Families of Groups and Vector Spaces

Anna Lango
Warsaw University
Białystok

Grzegorz Bancerek
Polish Academy of Sciences
Institute of Mathematics
Warsaw

Summary. In the first section we present properties of fields and Abelian groups in terms of commutativity, associativity, etc. Next, we are concerned with operations on n-tuples on some set which are generalization of operations on this set. It is used in third section to introduce the n-power of a group and the n-power of a field. Besides, we introduce a concept of indexed family of binary (unary) operations over some indexed family of sets and a product of such families which is binary (unary) operation on a product of family sets. We use that product in the last section to introduce the product of a finite sequence of Abelian groups.

MML Identifier: `PRVECT_1`.

The notation and terminology used in this paper are introduced in the following articles: [16], [9], [10], [13], [3], [17], [2], [5], [6], [12], [4], [8], [7], [14], [1], [11], and [15].

1. Abelian Groups and Fields

In the sequel G will denote an Abelian group. The following propositions are true:

(1) The addition of G is commutative.

(2) The addition of G is associative.

(3) The zero of G is a unity w.r.t. the addition of G.

(4) The reverse-map of G is an inverse operation w.r.t. the addition of G.

In the sequel G_1 will be a group structure. Next we state the proposition

ISSN 0777–4028

(5) If the addition of G_1 is commutative and the addition of G_1 is associative and the zero of G_1 is a unity w.r.t. the addition of G_1 and the reverse-map of G_1 is an inverse operation w.r.t. the addition of G_1, then G_1 is an Abelian group.

In the sequel F is a field. We now state several propositions:

(6) The addition of F is commutative.

(7) The multiplication of F is commutative.

(8) The addition of F is associative.

(9) The multiplication of F is associative.

(10) The zero of F is a unity w.r.t. the addition of F.

(11) The unity of F is a unity w.r.t. the multiplication of F.

(12) The reverse-map of F is an inverse operation w.r.t. the addition of F.

(13) The multiplication of F is distributive w.r.t. the addition of F.

One can verify that every field-like field structure is Abelian group-like.

2. The n-Product of a Binary and a Unary Operation

For simplicity we follow a convention: F is a field, n is a natural number, D is a non-empty set, d is an element of D, B is a binary operation on D, and C is a unary operation on D. We now define three new functors. Let us consider D, n, and let F be a binary operation on D, and let x, y be elements of D^n. Then $F^\circ(x, y)$ is an element of D^n. Let D be a non-empty set, and let F be a binary operation on D, and let n be a natural number. The functor $\pi^n F$ yields a binary operation on D^n and is defined by:

(Def.1) for all elements x, y of D^n holds $(\pi^n F)(x, y) = F^\circ(x, y)$.

Let us consider D, and let F be a unary operation on D, and let us consider n. The functor $\pi^n F$ yields a unary operation on D^n and is defined as follows:

(Def.2) for every element x of D^n holds $(\pi^n F)(x) = F \cdot x$.

Let D be a non-empty set, and let us consider n, and let x be an element of D. Then $n \longmapsto x$ is an element of D^n. We introduce the functor $n \dot{\longmapsto} x$ as a synonym of $n \longmapsto x$.

The following four propositions are true:

(14) If B is commutative, then $\pi^n B$ is commutative.

(15) If B is associative, then $\pi^n B$ is associative.

(16) If d is a unity w.r.t. B, then $n \dot{\longmapsto} d$ is a unity w.r.t. $\pi^n B$.

(17) If B has a unity and B is associative and C is an inverse operation w.r.t. B, then $\pi^n C$ is an inverse operation w.r.t. $\pi^n B$.

3. The n-Power of a Group and of a Field

Let F be an Abelian group, and let us consider n. The functor F^n yielding a strict Abelian group is defined as follows:

(Def.3) $F^n = \langle(\text{the carrier of } F)^n, \pi^n(\text{the addition of } F), \pi^n(\text{the reverse-map of } F), n \longmapsto \text{the zero of } F$ **qua** an element of (the carrier of $F)^n\rangle$.

We now define two new functors. Let us consider F, n. The functor $\cdot^n_F$ yields a function from $[\![$ the carrier of F, (the carrier of $F)^n$ $]\!]$ into (the carrier of $F)^n$ and is defined by:

(Def.4) for every element x of F and for every element v of (the carrier of $F)^n$ holds $(\cdot^n_F)(x, v) = (\text{the multiplication of } F)^\circ(x, v)$.

Let us consider F, n. The n-dimension vector space over F yielding a strict vector space structure over F is defined as follows:

(Def.5) the group structure of the n-dimension vector space over $F = F^n$ and the multiplication of the n-dimension vector space over $F = \cdot^n_F$.

For simplicity we follow a convention: D will be a non-empty set, H, G will be binary operations on D, d will be an element of D, and t_1, t_2 will be elements of D^n. One can prove the following proposition

(18) If H is distributive w.r.t. G, then $H^\circ(d, G^\circ(t_1, t_2)) = G^\circ(H^\circ(d, t_1), H^\circ(d, t_2))$.

Let D be a non-empty set, and let n be a natural number, and let F be a binary operation on D, and let x be an element of D, and let v be an element of D^n. Then $F^\circ(x, v)$ is an element of D^n. Let us consider F, n. Then the n-dimension vector space over F is a strict vector space over F.

4. Sequences of Non-empty Sets

In the sequel x will be arbitrary. We now define two new attributes. A function is non-empty set yielding if:

(Def.6) $\emptyset \notin \operatorname{rng} \mathrm{it}$.

A set is constituted functions if:

(Def.7) if $x \in \mathrm{it}$, then x is a function.

One can check that there exists a non-empty non-empty set yielding finite sequence and there exists a non-empty constituted functions set.

Let F be a constituted functions non-empty set. We see that the element of F is a function. Let f be a non-empty set yielding function. Then $\prod f$ is a constituted functions non-empty set. A sequence of non-empty sets is a non-empty non-empty set yielding finite sequence.

Let a be a non-empty function. Then $\operatorname{dom} a$ is a non-empty set.

The scheme *NEFinSeqLambda* concerns a non-empty finite sequence $\mathcal{A}$ and a unary functor $\mathcal{F}$ and states that:

there exists a non-empty finite sequence p such that $\operatorname{len} p = \operatorname{len} \mathcal{A}$ and for every element i of $\operatorname{dom} \mathcal{A}$ holds $p(i) = \mathcal{F}(i)$
for all values of the parameters.

Let a be a non-empty set yielding non-empty function, and let i be an element of $\operatorname{dom} a$. Then $a(i)$ is a non-empty set. Let a be a non-empty set yielding non-empty function, and let f be an element of $\prod a$, and let i be an element of $\operatorname{dom} a$. Then $f(i)$ is an element of $a(i)$.

5. The Product of Families of Operations

In the sequel a will denote a sequence of non-empty sets, i will denote an element of $\operatorname{dom} a$, and p will denote a finite sequence. We now define two new modes. Let a be a non-empty set yielding non-empty function. A function is called a family of binary operations of a if:

(Def.8) $\operatorname{dom} \mathrm{it} = \operatorname{dom} a$ and for every element i of $\operatorname{dom} a$ holds $\mathrm{it}(i)$ is a binary operation on $a(i)$.

A function is said to be a family of unary operations of a if:

(Def.9) $\operatorname{dom} \mathrm{it} = \operatorname{dom} a$ and for every element i of $\operatorname{dom} a$ holds $\mathrm{it}(i)$ is a unary operation on $a(i)$.

Let us consider a. Note that every family of binary operations of a is finite sequence-like and every family of unary operations of a is finite sequence-like.

The following two propositions are true:

(19) p is a family of binary operations of a if and only if $\operatorname{len} p = \operatorname{len} a$ and for every i holds $p(i)$ is a binary operation on $a(i)$.

(20) p is a family of unary operations of a if and only if $\operatorname{len} p = \operatorname{len} a$ and for every i holds $p(i)$ is a unary operation on $a(i)$.

Let us consider a, and let b be a family of binary operations of a, and let us consider i. Then $b(i)$ is a binary operation on $a(i)$. Let us consider a, and let u be a family of unary operations of a, and let us consider i. Then $u(i)$ is a unary operation on $a(i)$. Let F be a constituted functions non-empty set, and let u be a unary operation on F, and let f be an element of F. Then $u(f)$ is an element of F.

In the sequel f is arbitrary. One can prove the following proposition

(21) For all unary operations d, d' on $\prod a$ if for every element f of $\prod a$ and for every element i of $\operatorname{dom} a$ holds $d(f)(i) = d'(f)(i)$, then $d = d'$.

We now state the proposition

(22) For every family u of unary operations of a holds $\operatorname{dom}_\kappa u(\kappa) = a$ and $\prod(\operatorname{rng}_\kappa u(\kappa)) \subseteq \prod a$.

Let us consider a, and let u be a family of unary operations of a. Then $\prod^\circ u$ is a unary operation on $\prod a$.

We now state the proposition

(23) For every family u of unary operations of a and for every element f of $\prod a$ and for every element i of $\operatorname{dom} a$ holds $(\prod^\circ u)(f)(i) = u(i)(f(i))$.

Let F be a constituted functions non-empty set, and let b be a binary operation on F, and let f, g be elements of F. Then $b(f, g)$ is an element of F.

The following proposition is true

(24) For all binary operations d, d' on $\prod a$ if for all elements f, g of $\prod a$ and for every element i of $\operatorname{dom} a$ holds $d(f, g)(i) = d'(f, g)(i)$, then $d = d'$.

In the sequel i will denote an element of $\operatorname{dom} a$. Let us consider a, and let b be a family of binary operations of a. The functor $\prod^\circ b$ yields a binary operation on $\prod a$ and is defined by:

(Def.10) for all elements f, g of $\prod a$ and for every element i of $\operatorname{dom} a$ holds $(\prod^\circ b)(f, g)(i) = b(i)(f(i), g(i))$.

The following propositions are true:

(25) For every family b of binary operations of a if for every i holds $b(i)$ is commutative, then $\prod^\circ b$ is commutative.

(26) For every family b of binary operations of a if for every i holds $b(i)$ is associative, then $\prod^\circ b$ is associative.

(27) For every family b of binary operations of a and for every element f of $\prod a$ if for every i holds $f(i)$ is a unity w.r.t. $b(i)$, then f is a unity w.r.t. $\prod^\circ b$.

(28) For every family b of binary operations of a and for every family u of unary operations of a if for every i holds $u(i)$ is an inverse operation w.r.t. $b(i)$ and $b(i)$ has a unity, then $\prod^\circ u$ is an inverse operation w.r.t. $\prod^\circ b$.

6. The Product of Families of Groups

We now define three new constructions. A function is Abelian group yielding if:

(Def.11) if $x \in \operatorname{rng} \mathrm{it}$, then x is an Abelian group.

One can check that there exists a non-empty Abelian group yielding finite sequence.

A sequence of groups is a non-empty Abelian group yielding finite sequence.

Let g be a sequence of groups, and let i be an element of $\operatorname{dom} g$. Then $g(i)$ is an Abelian group. Let g be a sequence of groups. The functor $\overline{g}$ yielding a sequence of non-empty sets is defined as follows:

(Def.12) $\operatorname{len} \overline{g} = \operatorname{len} g$ and for every element j of $\operatorname{dom} g$ holds $\overline{g}(j) =$ the carrier of $g(j)$.

In the sequel g is a sequence of groups and i is an element of $\operatorname{dom} \overline{g}$. We now define four new functors. Let us consider g, i. Then $g(i)$ is an Abelian group. Let us consider g. The functor $\langle +_{g_i} \rangle_i$ yields a family of binary operations of $\overline{g}$ and is defined by:

(Def.13) $\mathrm{len}(\langle +_{g_i} \rangle_i) = \mathrm{len}\,\overline{g}$ and for every i holds $\langle +_{g_i} \rangle_i(i) =$ the addition of $g(i)$.

The functor $\langle -_{g_i} \rangle_i$ yields a family of unary operations of $\overline{g}$ and is defined by:

(Def.14) $\mathrm{len}(\langle -_{g_i} \rangle_i) = \mathrm{len}\,\overline{g}$ and for every i holds $\langle -_{g_i} \rangle_i(i) =$ the reverse-map of $g(i)$.

The functor $\langle 0_{g_i} \rangle_i$ yields an element of $\prod \overline{g}$ and is defined by:

(Def.15) for every i holds $\langle 0_{g_i} \rangle_i(i) =$ the zero of $g(i)$.

Let G be a sequence of groups. The functor $\prod G$ yields a strict Abelian group and is defined by:

(Def.16) $\prod G = \langle \prod \overline{G}, \prod^{\circ}(\langle +_{G_i} \rangle_i), \prod^{\circ}(\langle -_{G_i} \rangle_i), \langle 0_{G_i} \rangle_i \rangle$.

References

[1] Grzegorz Bancerek. Cartesian product of functions. *Formalized Mathematics*, 2(**4**):547–552, 1991.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[5] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.
[6] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[7] Czesław Byliński. Binary operations applied to finite sequences. *Formalized Mathematics*, 1(**4**):643–649, 1990.
[8] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[9] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[10] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[11] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[12] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[13] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[14] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.
[15] Andrzej Trybulec. Semilattice operations on finite subsets. *Formalized Mathematics*, 1(**2**):369–376, 1990.
[16] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[17] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

Received December 29, 1992