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Finite Topological Spaces

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Summary. By borrowing the concept of neighborhood from the theory of topological space in continuous cases and extending it to a discrete case such as a space of lattice points we have defined such concepts as boundaries, closures, interiors, isolated points, and connected points as in the case of continuity. We have proved various properties which are satisfied by these concepts.

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The articles [15], [8], [2], [5], [16], [6], [14], [19], [10], [12], [17], [9], [11], [3], [4], [13], [7], [18], and [1] provide the notation and terminology for this paper. The scheme *Set_of_Elements* deals with a non-empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{F}(x) : \mathcal{P}[x]\}$, where x ranges over elements of $\mathcal{A}$, is a subset of $\mathcal{A}$

for all values of the parameters.

One can prove the following propositions:

(1) Let A be a set. Let f be a finite sequence of elements of 2^A. Then if for every natural number i such that $1 \leq i$ and $i < \operatorname{len} f$ holds $\pi_i f \subseteq \pi_{i+1} f$, then for all natural numbers i, j such that $i \leq j$ and $1 \leq i$ and $j \leq \operatorname{len} f$ holds $\pi_i f \subseteq \pi_j f$.

(2) Let A be a set. Let f be a finite sequence of elements of 2^A. Suppose for every natural number i such that $1 \leq i$ and $i < \operatorname{len} f$ holds $\pi_i f \subseteq \pi_{i+1} f$. Then for all natural numbers i, j such that $i < j$ and $1 \leq i$ and $j \leq \operatorname{len} f$ and $\pi_j f \subseteq \pi_i f$ and for every natural number k such that $i \leq k$ and $k \leq j$ holds $\pi_j f = \pi_k f$.

(3) For every set F such that F is finite and $F \neq \emptyset$ and for all sets B, C such that $B \in F$ and $C \in F$ holds $B \subseteq C$ or $C \subseteq B$ there exists a set m such that $m \in F$ and for every set C such that $C \in F$ holds $C \subseteq m$.

(4) For all sets x, A holds $x \subseteq A$ if and only if $x \in 2^A$.

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(5) For every function f if for every natural number i holds $f(i) \subseteq f(i+1)$, and for all natural numbers i, j such that $i \leq j$ holds $f(i) \subseteq f(j)$.

The scheme *MaxFinSeqEx* deals with a non-empty set $\mathcal{A}$, a subset $\mathcal{B}$ of $\mathcal{A}$, a subset $\mathcal{C}$ of $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding a subset of $\mathcal{A}$ and states that:

there exists a finite sequence f of elements of $2^{\mathcal{A}}$ such that $\operatorname{len} f > 0$ and $\pi_1 f = \mathcal{C}$ and for every natural number i such that $i > 0$ and $i < \operatorname{len} f$ holds $\pi_{i+1} f = \mathcal{F}(\pi_i f)$ and $\mathcal{F}(\pi_{\operatorname{len} f} f) = \pi_{\operatorname{len} f} f$ and for all natural numbers i, j such that $i > 0$ and $i < j$ and $j \leq \operatorname{len} f$ holds $\pi_i f \subseteq \mathcal{B}$ and $\pi_i f \subseteq \pi_j f$ and $\pi_i f \neq \pi_j f$

provided the parameters meet the following requirements:

- $\mathcal{B}$ is finite,
- $\mathcal{C} \subseteq \mathcal{B}$,
- for every subset A of $\mathcal{A}$ such that $A \subseteq \mathcal{B}$ holds $A \subseteq \mathcal{F}(A)$ and $\mathcal{F}(A) \subseteq \mathcal{B}$.

We consider finite topology spaces which are extension of a 1-sorted structure and are systems

$\langle$a carrier, a neighbour-map$\rangle$,

where the carrier is a non-empty set and the neighbour-map is a function from the carrier into $2^{\text{the carrier}}$.

In the sequel F_1 denotes a finite topology space. We now define two new modes. Let F_1 be a 1-sorted structure. An element of F_1 is an element of the carrier of F_1.

A subset of F_1 is a subset of the carrier of F_1.

In the sequel x, y are elements of F_1. Let F_1 be a finite topology space, and let x be an element of F_1. The functor $U(x)$ yields a subset of F_1 and is defined as follows:

(Def.1) $U(x) = (\text{the neighbour-map of } F_1)(x)$.

One can prove the following proposition

(6) For every F_1 being a finite topology space and for every element x of F_1 holds $U(x) = (\text{the neighbour-map of } F_1)(x)$.

We now define three new constructions. Let x be arbitrary, and let y be a subset of $\{x\}$. Then $x \dot{\longmapsto} y$ is a function from $\{x\}$ into $2^{\{x\}}$. The strict finite topology space $\mathrm{FT}_{\{0\}}$ is defined as follows:

(Def.2) $\mathrm{FT}_{\{0\}} = \langle \{0 \,\mathbf{qua}\, \text{any}\}, 0 \dot{\longmapsto} \Omega_{\{0 \,\mathbf{qua}\, any\}} \rangle$.

A finite topology space is filled if:

(Def.3) for every element x of it holds $x \in U(x)$.

A 1-sorted structure is finite if:

(Def.4) the carrier of it is finite.

One can prove the following two propositions:

(7) $\mathrm{FT}_{\{0\}}$ is filled.

(8) $\mathrm{FT}_{\{0\}}$ is finite.

Let us observe that there exists a finite filled strict finite topology space.

Let T be a 1-sorted structure, and let F be a set. We say that F is a cover of T if and only if:

(Def.5) the carrier of $T \subseteq \bigcup F$.

Next we state the proposition

(9) For every F_1 being a filled finite topology space holds $\{U(x)\}$, where x ranges over elements of F_1, is a cover of F_1.

In the sequel A is a subset of F_1. Let us consider F_1, and let A be a subset of F_1. The functor A^δ yielding a subset of F_1 is defined as follows:

(Def.6) $A^\delta = \{x : U(x) \cap A \neq \emptyset \wedge U(x) \cap A^c \neq \emptyset\}$.

The following proposition is true

(10) $x \in A^\delta$ if and only if $U(x) \cap A \neq \emptyset$ and $U(x) \cap A^c \neq \emptyset$.

We now define two new functors. Let us consider F_1, and let A be a subset of F_1. The functor A^{δ_i} yielding a subset of F_1 is defined as follows:

(Def.7) $A^{\delta_i} = A \cap A^\delta$.

The functor A^{δ_o} yields a subset of F_1 and is defined as follows:

(Def.8) $A^{\delta_o} = A^c \cap A^\delta$.

Next we state the proposition

(11) $A^\delta = A^{\delta_i} \cup A^{\delta_o}$.

We now define several new constructions. Let us consider F_1, and let A be a subset of F_1. The functor A^i yielding a subset of F_1 is defined by:

(Def.9) $A^i = \{x : U(x) \subseteq A\}$.

The functor A^b yielding a subset of F_1 is defined as follows:

(Def.10) $A^b = \{x : U(x) \cap A \neq \emptyset\}$.

The functor A^s yielding a subset of F_1 is defined by:

(Def.11) $A^s = \{x : x \in A \wedge (U(x) \setminus \{x\}) \cap A = \emptyset\}$.

Let us consider F_1, and let A be a subset of F_1. The functor A^n yielding a subset of F_1 is defined as follows:

(Def.12) $A^n = A \setminus A^s$.

The functor A^f yields a subset of F_1 and is defined as follows:

(Def.13) $A^f = \{x : \bigvee_y [y \in A \wedge x \in U(y)]\}$.

A finite topology space is symmetric if:

(Def.14) for all elements x, y of the carrier of it such that $y \in U(x)$ holds $x \in U(y)$.

The following propositions are true:

(12) $x \in A^i$ if and only if $U(x) \subseteq A$.

(13) $x \in A^b$ if and only if $U(x) \cap A \neq \emptyset$.

(14) $x \in A^s$ if and only if $x \in A$ and $(U(x) \setminus \{x\}) \cap A = \emptyset$.

(15) $x \in A^n$ if and only if $x \in A$ and $(U(x) \setminus \{x\}) \cap A \neq \emptyset$.

(16) $x \in A^f$ if and only if there exists y such that $y \in A$ and $x \in U(y)$.

(17) F_1 is symmetric if and only if for every A holds $A^b = A^f$.

In the sequel F will be a subset of F_1. We now define five new constructions.

Let us consider F_1. A subset of F_1 is open if:

(Def.15) it $=$ it^i.

A subset of F_1 is closed if:

(Def.16) it $=$ it^b.

A subset of F_1 is connected if:

(Def.17) for all subsets B, C of F_1 such that it $= B \cup C$ and $B \neq \emptyset$ and $C \neq \emptyset$ and $B \cap C = \emptyset$ holds $B^b \cap C \neq \emptyset$.

Let us consider F_1, and let A be a subset of F_1. The functor A^{f_b} yields a subset of F_1 and is defined as follows:

(Def.18) $A^{f_b} = \bigcap\{F : A \subseteq F \wedge F \text{ is closed}\}$.

The functor A^{f_i} yielding a subset of F_1 is defined by:

(Def.19) $A^{f_i} = \bigcup\{F : A \subseteq F \wedge F \text{ is open}\}$.

Next we state a number of propositions:

(18) For every F_1 being a filled finite topology space and for every subset A of F_1 holds $A \subseteq A^b$.

(19) For every F_1 being a finite topology space and for all subsets A, B of F_1 such that $A \subseteq B$ holds $A^b \subseteq B^b$.

(20) Let F_1 be a filled finite finite topology space. Let A be a subset of F_1. Then A is connected if and only if for every element x of F_1 such that $x \in A$ there exists a finite sequence S of elements of $2^{\text{the carrier of } F_1}$ such that $\operatorname{len} S > 0$ and $\pi_1 S = \{x\}$ and for every natural number i such that $i > 0$ and $i < \operatorname{len} S$ holds $\pi_{i+1} S = (\pi_i S)^b \cap A$ and $A \subseteq \pi_{\operatorname{len} S} S$.

(21) For every non-empty set E and for every subset A of E and for every element x of E holds $x \in A^{\mathrm{c}}$ if and only if $x \notin A$.

(22) $((A^{\mathrm{c}})^i)^{\mathrm{c}} = A^b$.

(23) $((A^{\mathrm{c}})^b)^{\mathrm{c}} = A^i$.

(24) $A^\delta = A^b \cap (A^{\mathrm{c}})^b$.

(25) $(A^{\mathrm{c}})^\delta = A^\delta$.

(26) If $x \in A^s$, then $x \notin (A \setminus \{x\})^b$.

(27) If $A^s \neq \emptyset$ and $\operatorname{card} A > 1$, then A is connected.

(28) For every F_1 being a filled finite topology space and for every subset A of F_1 holds $A^i \subseteq A$.

(29) For every set E and for all subsets A, B of E holds $A = B$ if and only if $A^{\mathrm{c}} = B^{\mathrm{c}}$.

(30) If A is open, then A^{c} is closed.

(31) If A is closed, then A^{c} is open.

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