FORMALIZED MATHEMATICS
Vol.2,No.**5**, November–December 1991
Université Catholique de Louvain

Separated and Weakly Separated Subspaces of Topological Spaces

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Summary. A new concept of weakly separated subsets and subspaces of topological spaces is described in Mizar formalizm. Based on [1], in comparison with the notion of separated subsets (subspaces), some properties of such subsets (subspaces) are discussed. Some necessary facts concerning closed subspaces, open subspaces and the union and the meet of two subspaces are also introduced. To present the main theorems we first formulate basic definitions. Let X be a topological space. Two subsets A_1 and A_2 of X are called *weakly separated* if $A_1 \setminus A_2$ and $A_2 \setminus A_1$ are separated. Two subspaces X_1 and X_2 of X are called *weakly separated* if their carriers are weakly separated. The following theorem contains a useful characterization of weakly separated subsets in the special case when $A_1 \cup A_2$ is equal to the carrier of X. *A_1 and A_2 are weakly separated iff there are such subsets of X, C_1 and C_2 closed (open) and C open (closed, respectively), that $A_1 \cup A_2 = C_1 \cup C_2 \cup C$, $C_1 \subset A_1$, $C_2 \subset A_2$ and $C \subset A_1 \cap A_2$.* Next theorem divided into two parts contains similar characterization of weakly separated subspaces in the special case when the union of X_1 and X_2 is equal to X. *If X_1 meets X_2, then X_1 and X_2 are weakly separated iff either X_1 is a subspace of X_2 or X_2 is a subspace of X_1 or there are such open (closed) subspaces Y_1 and Y_2 of X that Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2 and either X is equal to the union of Y_1 and Y_2 or there is a(n) closed (open, respectively) subspace Y of X being a subspace of the meet of X_1 and X_2 and with the property that X is the union of all Y_1, Y_2 and Y. If X_1 misses X_2, then X_1 and X_2 are weakly separated iff X_1 and X_2 are open (closed) subspaces of X.* Moreover, the following simple characterization of separated subspaces by means of weakly separated ones is obtained. *X_1 and X_2 are separated iff there are weakly separated subspaces Y_1 and Y_2 of X such that X_1 is a subspace of Y_1, X_2 is a subspace of Y_2 and either Y_1 misses Y_2 or the meet of Y_1 and Y_2 misses the union of X_1 and X_2.*

MML Identifier: `TSEP_1`.

The papers [6], [7], [4], [3], [8], [2], and [5] provide the notation and terminology for this paper.

ISSN 0777–4028

1. Some properties of subspaces of topological spaces

In the sequel X is a topological space. We now state a number of propositions:

(1) For every subspace X_0 of X holds the carrier of X_0 is a subset of X.

(2) X is a subspace of X.

(3) For every strict topological space X holds $X \restriction \Omega_X = X$.

(4) For all subspaces X_1, X_2 of X holds the carrier of $X_1 \subseteq$ the carrier of X_2 if and only if X_1 is a subspace of X_2.

(5) For all strict subspaces X_1, X_2 of X holds the carrier of X_1 = the carrier of X_2 if and only if $X_1 = X_2$.

(6) For all strict subspaces X_1, X_2 of X holds X_1 is a subspace of X_2 and X_2 is a subspace of X_1 if and only if $X_1 = X_2$.

(7) For every subspace X_1 of X and for every subspace X_2 of X_1 holds X_2 is a subspace of X.

(8) For every subspace X_0 of X and for all subsets C, A of X and for every subset B of X_0 such that C is closed and $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$ holds B is closed if and only if A is closed.

(9) For every subspace X_0 of X and for all subsets C, A of X and for every subset B of X_0 such that C is open and $C \subseteq$ the carrier of X_0 and $A \subseteq C$ and $A = B$ holds B is open if and only if A is open.

(10) For every non-empty subset A_0 of X there exists a strict subspace X_0 of X such that $A_0 =$ the carrier of X_0.

(11) For every subspace X_0 of X and for every subset A of X such that $A =$ the carrier of X_0 holds X_0 is a closed subspace of X if and only if A is closed.

(12) For every closed subspace X_0 of X and for every subset A of X and for every subset B of X_0 such that $A = B$ holds B is closed if and only if A is closed.

(13) For every closed subspace X_1 of X and for every closed subspace X_2 of X_1 holds X_2 is a closed subspace of X.

(14) For every closed subspace X_1 of X and for every subspace X_2 of X such that the carrier of $X_1 \subseteq$ the carrier of X_2 holds X_1 is a closed subspace of X_2.

(15) For every non-empty subset A_0 of X such that A_0 is closed there exists a strict closed subspace X_0 of X such that $A_0 =$ the carrier of X_0.

Let X be a topological space. A subspace of X is said to be an open subspace of X if:

(Def.1) for every subset A of X such that $A =$ the carrier of it holds A is open.

The following propositions are true:

(16) For every subspace X_0 of X and for every subset A of X such that $A =$ the carrier of X_0 holds X_0 is an open subspace of X if and only if A is open.

(17) For every open subspace X_0 of X and for every subset A of X and for every subset B of X_0 such that $A = B$ holds B is open if and only if A is open.

(18) For every open subspace X_1 of X and for every open subspace X_2 of X_1 holds X_2 is an open subspace of X.

(19) For every open subspace X_1 of X and for every subspace X_2 of X such that the carrier of $X_1 \subseteq$ the carrier of X_2 holds X_1 is an open subspace of X_2.

(20) For every non-empty subset A_0 of X such that A_0 is open there exists a strict open subspace X_0 of X such that $A_0 =$ the carrier of X_0.

2. Operations on subspaces of topological spaces

In the sequel X denotes a topological space. Let us consider X, and let X_1, X_2 be subspaces of X. The functor $X_1 \cup X_2$ yielding a strict subspace of X is defined by:

(Def.2) the carrier of $X_1 \cup X_2 =$ (the carrier of X_1) $\cup$ (the carrier of X_2).

In the sequel X_1, X_2, X_3 will denote subspaces of X. One can prove the following propositions:

(21) $X_1 \cup X_2 = X_2 \cup X_1$ and $(X_1 \cup X_2) \cup X_3 = X_1 \cup (X_2 \cup X_3)$.

(22) X_1 is a subspace of $X_1 \cup X_2$ and X_2 is a subspace of $X_1 \cup X_2$.

(23) For all strict subspaces X_1, X_2 of X holds X_1 is a subspace of X_2 if and only if $X_1 \cup X_2 = X_2$ but X_2 is a subspace of X_1 if and only if $X_1 \cup X_2 = X_1$.

(24) For all closed subspaces X_1, X_2 of X holds $X_1 \cup X_2$ is a closed subspace of X.

(25) For all open subspaces X_1, X_2 of X holds $X_1 \cup X_2$ is an open subspace of X.

We now define two new predicates. Let us consider X, and let X_1, X_2 be subspaces of X. We say that X_1 misses X_2 if and only if:

(Def.3) (the carrier of X_1) $\cap$ (the carrier of X_2) $= \emptyset$.

We say that X_1 meets X_2 if and only if:

(Def.4) (the carrier of X_1) $\cap$ (the carrier of X_2) $\neq \emptyset$.

The following three propositions are true:

(26) X_1 misses X_2 if and only if X_1 does not meet X_2.

(27) X_1 misses X_2 if and only if X_2 misses X_1 but X_1 meets X_2 if and only if X_2 meets X_1.

(28) For all subsets A_1, A_2 of X such that A_1 = the carrier of X_1 and A_2 = the carrier of X_2 holds X_1 misses X_2 if and only if A_1 misses A_2 but X_1 meets X_2 if and only if A_1 meets A_2.

Let us consider X, and let X_1, X_2 be subspaces of X. Let us assume that X_1 meets X_2. The functor $X_1 \cap X_2$ yielding a strict subspace of X is defined by:

(Def.5) the carrier of $X_1 \cap X_2$ = (the carrier of X_1) $\cap$ (the carrier of X_2).

In the sequel X_1, X_2, X_3 will denote subspaces of X. We now state several propositions:

(29) If X_1 meets X_2 or X_2 meets X_1, then $X_1 \cap X_2 = X_2 \cap X_1$ but if X_1 meets X_2 and $X_1 \cap X_2$ meets X_3 or X_2 meets X_3 and X_1 meets $X_2 \cap X_3$, then $(X_1 \cap X_2) \cap X_3 = X_1 \cap (X_2 \cap X_3)$.

(30) If X_1 meets X_2, then $X_1 \cap X_2$ is a subspace of X_1 and $X_1 \cap X_2$ is a subspace of X_2.

(31) For all strict subspaces X_1, X_2 of X such that X_1 meets X_2 holds X_1 is a subspace of X_2 if and only if $X_1 \cap X_2 = X_1$ but X_2 is a subspace of X_1 if and only if $X_1 \cap X_2 = X_2$.

(32) For all closed subspaces X_1, X_2 of X such that X_1 meets X_2 holds $X_1 \cap X_2$ is a closed subspace of X.

(33) For all open subspaces X_1, X_2 of X such that X_1 meets X_2 holds $X_1 \cap X_2$ is an open subspace of X.

(34) If X_1 meets X_2, then $X_1 \cap X_2$ is a subspace of $X_1 \cup X_2$.

(35) For every subspace Y of X such that X_1 meets Y or Y meets X_1 but X_2 meets Y or Y meets X_2 holds $(X_1 \cup X_2) \cap Y = X_1 \cap Y \cup X_2 \cap Y$ and $Y \cap (X_1 \cup X_2) = Y \cap X_1 \cup Y \cap X_2$.

(36) For every subspace Y of X such that X_1 meets X_2 holds $X_1 \cap X_2 \cup Y = (X_1 \cup Y) \cap (X_2 \cup Y)$ and $Y \cup X_1 \cap X_2 = (Y \cup X_1) \cap (Y \cup X_2)$.

3. Separated and weakly separated subsets of topological spaces

Let X be a topological space, and let A_1, A_2 be subsets of X. Let us note that one can characterize the predicate A_1 and A_2 are separated by the following (equivalent) condition:

(Def.6) $\overline{A_1} \cap A_2 = \emptyset$ and $A_1 \cap \overline{A_2} = \emptyset$.

In the sequel X is a topological space and A_1, A_2 are subsets of X. We now state a number of propositions:

(37) If A_1 and A_2 are separated, then A_1 misses A_2.

(38) If A_1 is closed and A_2 is closed, then A_1 misses A_2 if and only if A_1 and A_2 are separated.

(39) If $A_1 \cup A_2$ is closed and A_1 and A_2 are separated, then A_1 is closed and A_2 is closed.

(40) If A_1 misses A_2, then if A_1 is open, then A_1 misses $\overline{A_2}$ but if A_2 is open, then $\overline{A_1}$ misses A_2.

(41) If A_1 is open and A_2 is open, then A_1 misses A_2 if and only if A_1 and A_2 are separated.

(42) If $A_1 \cup A_2$ is open and A_1 and A_2 are separated, then A_1 is open and A_2 is open.

(43) For every subset C of X such that A_1 and A_2 are separated holds $A_1 \cap C$ and $A_2 \cap C$ are separated and $C \cap A_1$ and $C \cap A_2$ are separated.

(44) For every subset B of X holds if A_1 and B are separated or A_2 and B are separated, then $A_1 \cap A_2$ and B are separated but if B and A_1 are separated or B and A_2 are separated, then B and $A_1 \cap A_2$ are separated.

(45) For every subset B of X holds A_1 and B are separated and A_2 and B are separated if and only if $A_1 \cup A_2$ and B are separated but B and A_1 are separated and B and A_2 are separated if and only if B and $A_1 \cup A_2$ are separated.

(46) A_1 and A_2 are separated if and only if there exist subsets C_1, C_2 of X such that $A_1 \subseteq C_1$ and $A_2 \subseteq C_2$ and C_1 misses A_2 and C_2 misses A_1 and C_1 is closed and C_2 is closed.

(47) A_1 and A_2 are separated if and only if there exist subsets C_1, C_2 of X such that $A_1 \subseteq C_1$ and $A_2 \subseteq C_2$ and $C_1 \cap C_2$ misses $A_1 \cup A_2$ and C_1 is closed and C_2 is closed.

(48) A_1 and A_2 are separated if and only if there exist subsets C_1, C_2 of X such that $A_1 \subseteq C_1$ and $A_2 \subseteq C_2$ and C_1 misses A_2 and C_2 misses A_1 and C_1 is open and C_2 is open.

(49) A_1 and A_2 are separated if and only if there exist subsets C_1, C_2 of X such that $A_1 \subseteq C_1$ and $A_2 \subseteq C_2$ and $C_1 \cap C_2$ misses $A_1 \cup A_2$ and C_1 is open and C_2 is open.

Let X be a topological space, and let A_1, A_2 be subsets of X. We say that A_1 and A_2 are weakly separated if and only if:

(Def.7) $A_1 \setminus A_2$ and $A_2 \setminus A_1$ are separated.

In the sequel X will be a topological space and A_1, A_2 will be subsets of X. We now state a number of propositions:

(50) If A_1 and A_2 are weakly separated, then A_2 and A_1 are weakly separated.

(51) A_1 misses A_2 and A_1 and A_2 are weakly separated if and only if A_1 and A_2 are separated.

(52) If $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$, then A_1 and A_2 are weakly separated.

(53) If A_1 is closed and A_2 is closed, then A_1 and A_2 are weakly separated.

(54) If A_1 is open and A_2 is open, then A_1 and A_2 are weakly separated.

(55) For every subset C of X such that A_1 and A_2 are weakly separated holds $A_1 \cup C$ and $A_2 \cup C$ are weakly separated and $C \cup A_1$ and $C \cup A_2$ are weakly separated.

(56) For all subsets B_1, B_2 of X such that $B_1 \subseteq A_2$ and $B_2 \subseteq A_1$ holds if A_1 and A_2 are weakly separated, then $A_1 \cup B_1$ and $A_2 \cup B_2$ are weakly separated and $B_1 \cup A_1$ and $B_2 \cup A_2$ are weakly separated.

(57) For every subset B of X holds if A_1 and B are weakly separated and A_2 and B are weakly separated, then $A_1 \cap A_2$ and B are weakly separated but if B and A_1 are weakly separated and B and A_2 are weakly separated, then B and $A_1 \cap A_2$ are weakly separated.

(58) For every subset B of X holds if A_1 and B are weakly separated and A_2 and B are weakly separated, then $A_1 \cup A_2$ and B are weakly separated but if B and A_1 are weakly separated and B and A_2 are weakly separated, then B and $A_1 \cup A_2$ are weakly separated.

(59) A_1 and A_2 are weakly separated if and only if there exist subsets C_1, C_2, C of X such that $C_1 \cap (A_1 \cup A_2) \subseteq A_1$ and $C_2 \cap (A_1 \cup A_2) \subseteq A_2$ and $C \cap (A_1 \cup A_2) \subseteq A_1 \cap A_2$ and the carrier of $X = C_1 \cup C_2 \cup C$ and C_1 is closed and C_2 is closed and C is open.

(60) Suppose A_1 and A_2 are weakly separated and $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$. Then there exist non-empty subsets C_1, C_2 of X such that C_1 is closed and C_2 is closed and $C_1 \cap (A_1 \cup A_2) \subseteq A_1$ and $C_2 \cap (A_1 \cup A_2) \subseteq A_2$ but $A_1 \cup A_2 \subseteq C_1 \cup C_2$ or there exists a non-empty subset C of X such that C is open and $C \cap (A_1 \cup A_2) \subseteq A_1 \cap A_2$ and the carrier of $X = C_1 \cup C_2 \cup C$.

(61) If $A_1 \cup A_2 =$ the carrier of X, then A_1 and A_2 are weakly separated if and only if there exist subsets C_1, C_2, C of X such that $A_1 \cup A_2 = C_1 \cup C_2 \cup C$ and $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ and $C \subseteq A_1 \cap A_2$ and C_1 is closed and C_2 is closed and C is open.

(62) Suppose $A_1 \cup A_2 =$ the carrier of X and A_1 and A_2 are weakly separated and $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$. Then there exist non-empty subsets C_1, C_2 of X such that C_1 is closed and C_2 is closed and $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ but $A_1 \cup A_2 = C_1 \cup C_2$ or there exists a non-empty subset C of X such that $A_1 \cup A_2 = C_1 \cup C_2 \cup C$ and $C \subseteq A_1 \cap A_2$ and C is open.

(63) A_1 and A_2 are weakly separated if and only if there exist subsets C_1, C_2, C of X such that $C_1 \cap (A_1 \cup A_2) \subseteq A_1$ and $C_2 \cap (A_1 \cup A_2) \subseteq A_2$ and $C \cap (A_1 \cup A_2) \subseteq A_1 \cap A_2$ and the carrier of $X = C_1 \cup C_2 \cup C$ and C_1 is open and C_2 is open and C is closed.

(64) Suppose A_1 and A_2 are weakly separated and $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$. Then there exist non-empty subsets C_1, C_2 of X such that C_1 is open and C_2 is open and $C_1 \cap (A_1 \cup A_2) \subseteq A_1$ and $C_2 \cap (A_1 \cup A_2) \subseteq A_2$ but $A_1 \cup A_2 \subseteq C_1 \cup C_2$ or there exists a non-empty subset C of X such that C is closed and $C \cap (A_1 \cup A_2) \subseteq A_1 \cap A_2$ and the carrier of $X = C_1 \cup C_2 \cup C$.

(65) If $A_1 \cup A_2 =$ the carrier of X, then A_1 and A_2 are weakly separated if and only if there exist subsets C_1, C_2, C of X such that $A_1 \cup A_2 = C_1 \cup C_2 \cup C$ and $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ and $C \subseteq A_1 \cap A_2$ and C_1 is open and C_2 is open and C is closed.

(66) Suppose $A_1 \cup A_2 =$ the carrier of X and A_1 and A_2 are weakly separated

and $A_1 \not\subseteq A_2$ and $A_2 \not\subseteq A_1$. Then there exist non-empty subsets C_1, C_2 of X such that C_1 is open and C_2 is open and $C_1 \subseteq A_1$ and $C_2 \subseteq A_2$ but $A_1 \cup A_2 = C_1 \cup C_2$ or there exists a non-empty subset C of X such that $A_1 \cup A_2 = C_1 \cup C_2 \cup C$ and $C \subseteq A_1 \cap A_2$ and C is closed.

(67) A_1 and A_2 are separated if and only if there exist subsets B_1, B_2 of X such that B_1 and B_2 are weakly separated and $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$ and $B_1 \cap B_2$ misses $A_1 \cup A_2$.

4. Separated and weakly separated subspaces of topological spaces

In the sequel X is a topological space. Let us consider X, and let X_1, X_2 be subspaces of X. We say that X_1 and X_2 are separated if and only if:

(Def.8) for all subsets A_1, A_2 of X such that $A_1 =$ the carrier of X_1 and $A_2 =$ the carrier of X_2 holds A_1 and A_2 are separated.

In the sequel X_1, X_2 will denote subspaces of X. One can prove the following propositions:

(68) If X_1 and X_2 are separated, then X_1 misses X_2.

(69) If X_1 and X_2 are separated, then X_2 and X_1 are separated.

(70) For all closed subspaces X_1, X_2 of X holds X_1 misses X_2 if and only if X_1 and X_2 are separated.

(71) If $X = X_1 \cup X_2$ and X_1 and X_2 are separated, then X_1 is a closed subspace of X and X_2 is a closed subspace of X.

(72) If $X_1 \cup X_2$ is a closed subspace of X and X_1 and X_2 are separated, then X_1 is a closed subspace of X and X_2 is a closed subspace of X.

(73) For all open subspaces X_1, X_2 of X holds X_1 misses X_2 if and only if X_1 and X_2 are separated.

(74) If $X = X_1 \cup X_2$ and X_1 and X_2 are separated, then X_1 is an open subspace of X and X_2 is an open subspace of X.

(75) If $X_1 \cup X_2$ is an open subspace of X and X_1 and X_2 are separated, then X_1 is an open subspace of X and X_2 is an open subspace of X.

(76) For all subspaces Y, X_1, X_2 of X such that X_1 meets Y and X_2 meets Y holds if X_1 and X_2 are separated, then $X_1 \cap Y$ and $X_2 \cap Y$ are separated and $Y \cap X_1$ and $Y \cap X_2$ are separated.

(77) For all subspaces Y_1, Y_2 of X such that Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2 holds if X_1 and X_2 are separated, then Y_1 and Y_2 are separated.

(78) For every subspace Y of X such that X_1 meets X_2 holds if X_1 and Y are separated or X_2 and Y are separated, then $X_1 \cap X_2$ and Y are separated but if Y and X_1 are separated or Y and X_2 are separated, then Y and $X_1 \cap X_2$ are separated.

(79) For every subspace Y of X holds X_1 and Y are separated and X_2 and Y are separated if and only if $X_1 \cup X_2$ and Y are separated but Y and X_1 are separated and Y and X_2 are separated if and only if Y and $X_1 \cup X_2$ are separated.

(80) X_1 and X_2 are separated if and only if there exist closed subspaces Y_1, Y_2 of X such that X_1 is a subspace of Y_1 and X_2 is a subspace of Y_2 and Y_1 misses X_2 and Y_2 misses X_1.

(81) X_1 and X_2 are separated if and only if there exist closed subspaces Y_1, Y_2 of X such that X_1 is a subspace of Y_1 and X_2 is a subspace of Y_2 but Y_1 misses Y_2 or $Y_1 \cap Y_2$ misses $X_1 \cup X_2$.

(82) X_1 and X_2 are separated if and only if there exist open subspaces Y_1, Y_2 of X such that X_1 is a subspace of Y_1 and X_2 is a subspace of Y_2 and Y_1 misses X_2 and Y_2 misses X_1.

(83) X_1 and X_2 are separated if and only if there exist open subspaces Y_1, Y_2 of X such that X_1 is a subspace of Y_1 and X_2 is a subspace of Y_2 but Y_1 misses Y_2 or $Y_1 \cap Y_2$ misses $X_1 \cup X_2$.

Let X be a topological space, and let X_1, X_2 be subspaces of X. We say that X_1 and X_2 are weakly separated if and only if:

(Def.9) for all subsets A_1, A_2 of X such that $A_1 =$ the carrier of X_1 and $A_2 =$ the carrier of X_2 holds A_1 and A_2 are weakly separated.

In the sequel X_1, X_2 will denote subspaces of X. The following propositions are true:

(84) If X_1 and X_2 are weakly separated, then X_2 and X_1 are weakly separated.

(85) X_1 misses X_2 and X_1 and X_2 are weakly separated if and only if X_1 and X_2 are separated.

(86) If X_1 is a subspace of X_2 or X_2 is a subspace of X_1, then X_1 and X_2 are weakly separated.

(87) For all closed subspaces X_1, X_2 of X holds X_1 and X_2 are weakly separated.

(88) For all open subspaces X_1, X_2 of X holds X_1 and X_2 are weakly separated.

(89) For every subspace Y of X such that X_1 and X_2 are weakly separated holds $X_1 \cup Y$ and $X_2 \cup Y$ are weakly separated and $Y \cup X_1$ and $Y \cup X_2$ are weakly separated.

(90) For all subspaces Y_1, Y_2 of X such that Y_1 is a subspace of X_2 and Y_2 is a subspace of X_1 holds if X_1 and X_2 are weakly separated, then $X_1 \cup Y_1$ and $X_2 \cup Y_2$ are weakly separated and $Y_1 \cup X_1$ and $Y_2 \cup X_2$ are weakly separated.

(91) For all subspaces Y, X_1, X_2 of X such that X_1 meets X_2 holds if X_1 and Y are weakly separated and X_2 and Y are weakly separated, then $X_1 \cap X_2$

and Y are weakly separated but if Y and X_1 are weakly separated and Y and X_2 are weakly separated, then Y and $X_1 \cap X_2$ are weakly separated.

(92) For every subspace Y of X holds if X_1 and Y are weakly separated and X_2 and Y are weakly separated, then $X_1 \cup X_2$ and Y are weakly separated but if Y and X_1 are weakly separated and Y and X_2 are weakly separated, then Y and $X_1 \cup X_2$ are weakly separated.

(93) Let X be a strict topological space. Let X_1, X_2 be subspaces of X. Suppose X_1 meets X_2. Then X_1 and X_2 are weakly separated if and only if X_1 is a subspace of X_2 or X_2 is a subspace of X_1 or there exist closed subspaces Y_1, Y_2 of X such that $Y_1 \cap (X_1 \cup X_2)$ is a subspace of X_1 and $Y_2 \cap (X_1 \cup X_2)$ is a subspace of X_2 but $X_1 \cup X_2$ is a subspace of $Y_1 \cup Y_2$ or there exists an open subspace Y of X such that $X = Y_1 \cup Y_2 \cup Y$ and $Y \cap (X_1 \cup X_2)$ is a subspace of $X_1 \cap X_2$.

(94) Suppose $X = X_1 \cup X_2$ and X_1 meets X_2. Then X_1 and X_2 are weakly separated if and only if X_1 is a subspace of X_2 or X_2 is a subspace of X_1 or there exist closed subspaces Y_1, Y_2 of X such that Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2 but $X = Y_1 \cup Y_2$ or there exists an open subspace Y of X such that $X = Y_1 \cup Y_2 \cup Y$ and Y is a subspace of $X_1 \cap X_2$.

(95) If $X = X_1 \cup X_2$ and X_1 misses X_2, then X_1 and X_2 are weakly separated if and only if X_1 is a closed subspace of X and X_2 is a closed subspace of X.

(96) Let X be a strict topological space. Let X_1, X_2 be subspaces of X. Suppose X_1 meets X_2. Then X_1 and X_2 are weakly separated if and only if X_1 is a subspace of X_2 or X_2 is a subspace of X_1 or there exist open subspaces Y_1, Y_2 of X such that $Y_1 \cap (X_1 \cup X_2)$ is a subspace of X_1 and $Y_2 \cap (X_1 \cup X_2)$ is a subspace of X_2 but $X_1 \cup X_2$ is a subspace of $Y_1 \cup Y_2$ or there exists a closed subspace Y of X such that $X = Y_1 \cup Y_2 \cup Y$ and $Y \cap (X_1 \cup X_2)$ is a subspace of $X_1 \cap X_2$.

(97) Suppose $X = X_1 \cup X_2$ and X_1 meets X_2. Then X_1 and X_2 are weakly separated if and only if X_1 is a subspace of X_2 or X_2 is a subspace of X_1 or there exist open subspaces Y_1, Y_2 of X such that Y_1 is a subspace of X_1 and Y_2 is a subspace of X_2 but $X = Y_1 \cup Y_2$ or there exists a closed subspace Y of X such that $X = Y_1 \cup Y_2 \cup Y$ and Y is a subspace of $X_1 \cap X_2$.

(98) If $X = X_1 \cup X_2$ and X_1 misses X_2, then X_1 and X_2 are weakly separated if and only if X_1 is an open subspace of X and X_2 is an open subspace of X.

(99) X_1 and X_2 are separated if and only if there exist subspaces Y_1, Y_2 of X such that Y_1 and Y_2 are weakly separated and X_1 is a subspace of Y_1 and X_2 is a subspace of Y_2 but Y_1 misses Y_2 or $Y_1 \cap Y_2$ misses $X_1 \cup X_2$.

Acknowledgements

I would like to express my gratitude to Professors A. Trybulec and C. Byliński for their valuable advice. I am also extremely grateful to W.A. Trybulec and to all participants of the Mizar Summer School (1991) for acquainting me with the Mizar system.

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Received January 8, 1992