FORMALIZED MATHEMATICS
Vol.2,No.5, November–December 1991
Université Catholique de Louvain

Complete Lattices

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Summary. In the first section the lattice of subsets of distinct set is introduced. The join and meet operations are, respectively, union and intersection of sets, and the ordering relation is inclusion. It is shown that this lattice is Boolean, i.e. distributive and complimentary. The socond section introduced the poset generated in a distinct lattice by its ordering relation. Besides, it is proved that posets which have l.u.b.'s and g.l.b.'s for every two elements generate lattices with the same ordering relations. In the last section the concept of complete lattice is introduced and discussed. Finally, the fact that the function f from subsets of distinct set yielding elements of this set is a infinite union of some complete lattice, if f yields an element a for singleton $\{a\}$ and $f(f^{\circ}X) = f(\bigsqcup X)$ for every subset X, is proved. Some concepts and proofs are based on [6] and [7].

MML Identifier: `LATTICE3`.

The notation and terminology used here are introduced in the following articles: [10], [8], [13], [4], [5], [3], [17], [14], [15], [1], [9], [2], [16], [11], and [12].

1. Boolean lattice of subsets

Let X be a non-empty set, and let x, y be elements of X. Then $\{x,y\}$ is a non-empty subset of X.

Let X be a set, and let x, y be elements of 2^X. Then $x \cup y$ is a subset of X. Then $x \cap y$ is a subset of X.

Let X be a set. The lattice of subsets of X yields a strict lattice structure and is defined by:

(Def.1) the carrier of the lattice of subsets of $X = 2^X$ and for all elements Y, Z of 2^X holds (the join operation of the lattice of subsets of X)(Y, Z) $= Y \cup Z$ and (the meet operation of the lattice of subsets of X)(Y, Z) $= Y \cap Z$.

ISSN 0777–4028

In the sequel X will denote a set and x, y will denote elements of the lattice of subsets of X. The following propositions are true:

(1) $x \sqcup y = x \cup y$ and $x \sqcap y = x \cap y$.

(2) $x \sqsubseteq y$ if and only if $x \subseteq y$.

Let us consider X. Then the lattice of subsets of X is a strict lattice.

In the sequel x will denote an element of the lattice of subsets of X. The following propositions are true:

(3) The lattice of subsets of X is a lower bound lattice and $\perp_{\text{the lattice of subsets of } X} = \emptyset$.

(4) The lattice of subsets of X is an upper bound lattice and $\top_{\text{the lattice of subsets of } X} = X$.

Let us consider X. Then the lattice of subsets of X is a strict Boolean lattice.

Next we state the proposition

(5) For every element x of the lattice of subsets of X holds $x^{\mathrm{c}} = X \setminus x$.

2. Correspondence between lattices and posets

Let L be a lattice. Then $\mathrm{LattRel}(L)$ is an order in the carrier of L.

Let L be a lattice. The functor $\mathrm{Poset}(L)$ yields a strict poset and is defined as follows:

(Def.2) $\mathrm{Poset}(L) = \langle \text{the carrier of } L, \mathrm{LattRel}(L) \rangle$.

Next we state the proposition

(6) For all lattices L_1, L_2 such that $\mathrm{Poset}(L_1) = \mathrm{Poset}(L_2)$ holds the lattice structure of $L_1 =$ the lattice structure of L_2.

Let L be a lattice, and let p be an element of L. The functor $p^{\cdot}$ yields an element of $\mathrm{Poset}(L)$ and is defined as follows:

(Def.3) $p^{\cdot} = p$.

Let L be a lattice, and let p be an element of $\mathrm{Poset}(L)$. The functor ${}^{\cdot}p$ yielding an element of L is defined as follows:

(Def.4) ${}^{\cdot}p = p$.

In the sequel L is a lattice, p, q are elements of L, and p' is an element of $\mathrm{Poset}(L)$. We now state the proposition

(7) $p \sqsubseteq q$ if and only if $p^{\cdot} \leq q^{\cdot}$.

Let X be a set, and let O be an order in X. Then $O^{\smile}$ is an order in X.

Let A be a poset. The functor $A^{\smile}$ yields a strict poset and is defined as follows:

(Def.5) $A^{\smile} = \langle \text{the carrier of } A, (\text{the order of } A)^{\smile} \rangle$.

In the sequel A will be a poset and a, b, c will be elements of A. One can prove the following proposition

(8) $(A^{\smile})^{\smile} =$ the poset of A.

Let A be a poset, and let a be an element of A. The functor $a^{\smile}$ yielding an element of $A^{\smile}$ is defined as follows:

(Def.6) $a^{\smile} = a$.

Let A be a poset, and let a be an element of $A^{\smile}$. The functor $\curvearrowleft a$ yielding an element of A is defined by:

(Def.7) $\curvearrowleft a = a$.

One can prove the following proposition

(9) $a \leq b$ if and only if $b^{\smile} \leq a^{\smile}$.

We now define four new predicates. Let A be a poset, and let X be a set, and let a be an element of A. The predicate $a \leq X$ is defined as follows:

(Def.8) for every element b of A such that $b \in X$ holds $a \leq b$.

We write $X \geq a$ if $a \leq X$. The predicate $X \leq a$ is defined by:

(Def.9) for every element b of A such that $b \in X$ holds $b \leq a$.

We write $a \geq X$ if and only if $X \leq a$.

We now define two new attributes. A poset has l.u.b.'s if:

(Def.10) for every elements x, y of it there exists an element z of it such that $x \leq z$ and $y \leq z$ and for every element z' of it such that $x \leq z'$ and $y \leq z'$ holds $z \leq z'$.

A poset has g.l.b.'s if:

(Def.11) for every elements x, y of it there exists an element z of it such that $z \leq x$ and $z \leq y$ and for every element z' of it such that $z' \leq x$ and $z' \leq y$ holds $z' \leq z$.

We now state two propositions:

(10) A has l.u.b.'s if and only if $A^{\smile}$ has g.l.b.'s.

(11) For every lattice L holds Poset(L) has l.u.b.'s and g.l.b.'s.

A poset is complete if:

(Def.12) for every set X there exists an element a of it such that $X \leq a$ and for every element b of it such that $X \leq b$ holds $a \leq b$.

Next we state the proposition

(12) If A is complete, then A has l.u.b.'s and g.l.b.'s.

Let A be a poset satisfying the condition: A has l.u.b.'s. Let a, b be elements of A. The functor $a \sqcup b$ yielding an element of A is defined as follows:

(Def.13) $a \leq a \sqcup b$ and $b \leq a \sqcup b$ and for every element c of A such that $a \leq c$ and $b \leq c$ holds $a \sqcup b \leq c$.

Let A be a poset satisfying the condition: A has g.l.b.'s. Let a, b be elements of A. The functor $a \sqcap b$ yields an element of A and is defined as follows:

(Def.14) $a \sqcap b \leq a$ and $a \sqcap b \leq b$ and for every element c of A such that $c \leq a$ and $c \leq b$ holds $c \leq a \sqcap b$.

For simplicity we follow a convention: V denotes a poset with l.u.b.'s, u_1, u_2, u_3 denote elements of V, N denotes a poset with g.l.b.'s, n_1, n_2, n_3 denote elements of N, K denotes a poset with l.u.b.'s and g.l.b.'s, and k_1, k_2 denote elements of K. The following propositions are true:

(13) $u_1 \sqcup u_2 = u_2 \sqcup u_1$.

(14) $(u_1 \sqcup u_2) \sqcup u_3 = u_1 \sqcup (u_2 \sqcup u_3)$.

(15) $n_1 \sqcap n_2 = n_2 \sqcap n_1$.

(16) $(n_1 \sqcap n_2) \sqcap n_3 = n_1 \sqcap (n_2 \sqcap n_3)$.

(17) $k_1 \sqcap k_2 \sqcup k_2 = k_2$.

(18) $k_1 \sqcap (k_1 \sqcup k_2) = k_1$.

(19) For every A being a poset with l.u.b.'s and g.l.b.'s there exists a strict lattice L such that the poset of $A = \mathrm{Poset}(L)$.

Let us consider A satisfying the condition: A has l.u.b.'s and g.l.b.'s. The functor $\mathbb{L}_A$ yields a strict lattice and is defined as follows:

(Def.15) the poset of $A = \mathrm{Poset}(\mathbb{L}_A)$.

The following proposition is true

(20) $\mathrm{LattRel}(L)^\smile = \mathrm{LattRel}(L^\circ)$ and $\mathrm{Poset}(L)^\smile = \mathrm{Poset}(L^\circ)$.

3. Complete lattices

Let L be a lattice structure. A subset of L is a subset of the carrier of L.

We now define four new predicates. Let L be a lattice structure, and let p be an element of L, and let X be a set. The predicate $p \sqsubseteq X$ is defined by:

(Def.16) for every element q of L such that $q \in X$ holds $p \sqsubseteq q$.

We write $X \sqsupseteq p$ if $p \sqsubseteq X$. The predicate $X \sqsubseteq p$ is defined by:

(Def.17) for every element q of L such that $q \in X$ holds $q \sqsubseteq p$.

We write $p \sqsupseteq X$ if $X \sqsubseteq p$.

We now state two propositions:

(21) For every lattice L and for all elements p, q, r of L holds $p \sqsubseteq \{q, r\}$ if and only if $p \sqsubseteq q \sqcap r$.

(22) For every lattice L and for all elements p, q, r of L holds $p \sqsupseteq \{q, r\}$ if and only if $q \sqcup r \sqsubseteq p$.

We now define three new attributes. A lattice structure is complete if:

(Def.18) for every set X there exists an element p of it such that $X \sqsubseteq p$ and for every element r of it such that $X \sqsubseteq r$ holds $p \sqsubseteq r$.

A lattice structure is $\bigsqcup$-distributive if it satisfies the condition (Def.19).

(Def.19) Given X. Let a, b, c be elements of it. Then if $X \sqsubseteq a$ and for every element d of it such that $X \sqsubseteq d$ holds $a \sqsubseteq d$ and $\{b \sqcap a' : a' \in X\} \sqsubseteq c$, where a' ranges over elements of it and for every element d of it such that $\{b \sqcap a' : a' \in X\} \sqsubseteq d$, where a' ranges over elements of it holds $c \sqsubseteq d$, then $b \sqcap a \sqsubseteq c$.

A lattice structure is $\bigsqcap$-distributive if it satisfies the condition (Def.20).

(Def.20) Given X. Let a, b, c be elements of it. Then if $X \sqsupseteq a$ and for every element d of it such that $X \sqsupseteq d$ holds $d \sqsubseteq a$ and $\{b \sqcup a' : a' \in X\} \sqsupseteq c$, where a' ranges over elements of it and for every element d of it such that $\{b \sqcup a' : a' \in X\} \sqsupseteq d$, where a' ranges over elements of it holds $d \sqsubseteq c$, then $c \sqsubseteq b \sqcup a$.

We now state several propositions:

(23) For every Boolean lattice B and for every element a of B holds $X \sqsubseteq a$ if and only if $\{b^{\mathrm{c}} : b \in X\} \sqsupseteq a^{\mathrm{c}}$, where b ranges over elements of B.

(24) For every Boolean lattice B and for every element a of B holds $X \sqsupseteq a$ if and only if $\{b^{\mathrm{c}} : b \in X\} \sqsubseteq a^{\mathrm{c}}$, where b ranges over elements of B.

(25) The lattice of subsets of X is complete.

(26) The lattice of subsets of X is $\bigsqcup$-distributive.

(27) The lattice of subsets of X is $\bigsqcap$-distributive.

Next we state four propositions:

(28) $p \sqsubseteq X$ if and only if $p^{\cdot} \leq X$.

(29) $p' \leq X$ if and only if ${}^{\cdot}p' \sqsubseteq X$.

(30) $X \sqsubseteq p$ if and only if $X \leq p^{\cdot}$.

(31) $X \leq p'$ if and only if $X \sqsubseteq {}^{\cdot}p'$.

Let A be a complete poset. Then $\mathbb{L}_A$ is a complete strict lattice.

Let L be a lattice structure satisfying the condition: L is a complete lattice. Let X be a set. The functor $\bigsqcup_L X$ yields an element of L and is defined by:

(Def.21) $X \sqsubseteq \bigsqcup_L X$ and for every element r of L such that $X \sqsubseteq r$ holds $\bigsqcup_L X \sqsubseteq r$.

Let L be a lattice structure, and let X be a set. The functor $\bigsqcap_L X$ yielding an element of L is defined as follows:

(Def.22) $\bigsqcap_L X = \bigsqcup_L \{p : p \sqsubseteq X\}$, where p ranges over elements of L.

We now define two new functors. Let L be a lattice structure, and let X be a subset of L. We introduce the functor $\bigsqcup X$ as a synonym of $\bigsqcup_L X$. We introduce the functor $\bigsqcap X$ as a synonym of $\bigsqcap_L X$.

We adopt the following rules: C denotes a complete lattice, a, b, c denote elements of C, and X, Y denote sets. Next we state a number of propositions:

(32) $\bigsqcup_C \{a \sqcap b : b \in X\} \sqsubseteq a \sqcap \bigsqcup_C X$.

(33) C is $\bigsqcup$-distributive if and only if for all X, a holds $a \sqcap \bigsqcup_C X \sqsubseteq \bigsqcup_C \{a \sqcap b : b \in X\}$.

(34) $a = \bigsqcap_C X$ if and only if $a \sqsubseteq X$ and for every b such that $b \sqsubseteq X$ holds $b \sqsubseteq a$.

(35) $a \sqcup \bigsqcap_C X \sqsubseteq \bigsqcap_C \{a \sqcup b : b \in X\}$.

(36) C is $\bigsqcap$-distributive if and only if for all X, a holds $\bigsqcap_C \{a \sqcup b : b \in X\} \sqsubseteq a \sqcup \bigsqcap_C X$.

(37) $\bigsqcup_C X = \bigsqcap_C \{a : a \sqsupseteq X\}$.
(38) If $a \in X$, then $a \sqsubseteq \bigsqcup_C X$ and $\bigsqcap_C X \sqsubseteq a$.
(39) If $X \sqsubseteq a$, then $\bigsqcup_C X \sqsubseteq a$.
(40) If $a \sqsubseteq X$, then $a \sqsubseteq \bigsqcap_C X$.
(41) If $a \in X$ and $X \sqsubseteq a$, then $\bigsqcup_C X = a$.
(42) If $a \in X$ and $a \sqsubseteq X$, then $\bigsqcap_C X = a$.
(43) $\bigsqcup\{a\} = a$ and $\bigsqcap\{a\} = a$.
(44) $a \sqcup b = \bigsqcup\{a, b\}$ and $a \sqcap b = \bigsqcap\{a, b\}$.
(45) $a = \bigsqcup_C \{b : b \sqsubseteq a\}$ and $a = \bigsqcap_C \{c : a \sqsubseteq c\}$.
(46) If $X \subseteq Y$, then $\bigsqcup_C X \sqsubseteq \bigsqcup_C Y$ and $\bigsqcap_C Y \sqsubseteq \bigsqcap_C X$.
(47) $\bigsqcup_C X = \bigsqcup_C \{a : \bigvee_b [a \sqsubseteq b \wedge b \in X]\}$ and $\bigsqcap_C X = \bigsqcap_C \{b : \bigvee_a [a \sqsubseteq b \wedge a \in X]\}$.
(48) If for every a such that $a \in X$ there exists b such that $a \sqsubseteq b$ and $b \in Y$, then $\bigsqcup_C X \sqsubseteq \bigsqcup_C Y$.
(49) If $X \subseteq 2^{\text{the carrier of } C}$, then $\bigsqcup_C \bigcup X = \bigsqcup_C \{\bigsqcup Y : Y \in X\}$, where Y ranges over subsets of C.
(50) C is a lower bound lattice and $\bot_C = \bigsqcup_C \emptyset$.
(51) C is an upper bound lattice and $\top_C = \bigsqcup_C$ (the carrier of C).
(52) If C is an implicative lattice, then $a \Rightarrow b = \bigsqcup_C \{c : a \sqcap c \sqsubseteq b\}$.
(53) If C is an implicative lattice, then C is $\bigsqcup$-distributive.
(54) For every complete $\bigsqcup$-distributive lattice D and for every element a of D holds $a \sqcap \bigsqcup_D X = \bigsqcup_D \{a \sqcap b_1 : b_1 \in X\}$, where b_1 ranges over elements of D and $\bigsqcup_D X \sqcap a = \bigsqcup_D \{b_2 \sqcap a : b_2 \in X\}$, where b_2 ranges over elements of D.
(55) For every complete $\bigsqcap$-distributive lattice D and for every element a of D holds $a \sqcup \bigsqcap_D X = \bigsqcap_D \{a \sqcup b_1 : b_1 \in X\}$, where b_1 ranges over elements of D and $\bigsqcap_D X \sqcup a = \bigsqcap_D \{b_2 \sqcup a : b_2 \in X\}$, where b_2 ranges over elements of D.

In this article we present several logical schemes. The scheme *SingleFraenkel* deals with a constant $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

$\{\mathcal{A} : \mathcal{P}[a]\} = \{\mathcal{A}\}$, where a ranges over elements of $\mathcal{B}$

provided the parameters meet the following requirement:

- there exists an element a of $\mathcal{B}$ such that $\mathcal{P}[a]$.

The scheme *FuncFraenkel* deals with a non-empty set $\mathcal{A}$, a non-empty set $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$, a function $\mathcal{C}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{C} \circ \{\mathcal{F}(x) : \mathcal{P}[x]\} = \{\mathcal{C}(\mathcal{F}(x)) : \mathcal{P}[x]\}$, where x ranges over elements of $\mathcal{A}$, and x ranges over elements of $\mathcal{A}$

provided the parameters satisfy the following condition:

- $\mathcal{B} \subseteq \operatorname{dom} \mathcal{C}$.

The following proposition is true

(56) Let D be a non-empty set. Let f be a function from 2^D into D. Then if for every element a of D holds $f(\{a\}) = a$ and for every subset X of 2^D holds $f(f^\circ X) = f(\bigcup X)$, then there exists a complete strict lattice L such that the carrier of $L = D$ and for every subset X of L holds $\bigsqcup X = f(X)$.

References

[1] Grzegorz Bancerek. Filters - part I. *Formalized Mathematics*, 1(**5**):813–819, 1990.
[2] Grzegorz Bancerek. Filters - part II. Quotient lattices modulo filters and direct product of two lattices. *Formalized Mathematics*, 2(**3**):433–438, 1991.
[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Helena Rasiowa and Roman Sikorski. *The Mathematics of Metamathematics*. Volume 41 of *Monografie Matematyczne*, PWN, Warszawa, 1968.
[7] Tadeusz Traczyk. *Wstęp do teorii algebr Boole'a*. Volume 37 of *Biblioteka Matematyczna*, PWN, Warszawa, 1970.
[8] Andrzej Trybulec. Enumerated sets. *Formalized Mathematics*, 1(**1**):25–34, 1990.
[9] Andrzej Trybulec. Finite join and finite meet and dual lattices. *Formalized Mathematics*, 1(**5**):983–988, 1990.
[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[11] Wojciech A. Trybulec. Partially ordered sets. *Formalized Mathematics*, 1(**2**):313–319, 1990.
[12] Wojciech A. Trybulec and Grzegorz Bancerek. Kuratowski - Zorn lemma. *Formalized Mathematics*, 1(**2**):387–393, 1990.
[13] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[14] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[15] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[16] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Formalized Mathematics*, 1(**1**):85–89, 1990.
[17] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

Received May 13, 1992