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Totally Bounded Metric Spaces

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The papers [19], [9], [1], [4], [20], [2], [18], [13], [5], [8], [14], [21], [7], [15], [12], [11], [17], [6], [10], [16], and [3] provide the terminology and notation for this paper. For simplicity we follow the rules: M is a metric space, c, g are elements of the carrier of M, F is a family of subsets of the carrier of M, A, B are subsets of the carrier of M, f is a function, n, m, p, k are natural numbers, and r, s, L are real numbers. Next we state four propositions:

(1) For every L such that $0 < L$ and $L < 1$ for all n, m such that $n \leq m$ holds $L^m \leq L^n$.

(2) For every L such that $0 < L$ and $L < 1$ for every k holds $L^k \leq 1$ and $0 < L^k$.

(3) For every L such that $0 < L$ and $L < 1$ for every s such that $0 < s$ there exists n such that $L^n < s$.

(4) For every set X such that X is finite and $X \neq \emptyset$ and for all sets Y, Z such that $Y \in X$ and $Z \in X$ holds $Y \subseteq Z$ or $Z \subseteq Y$ there exists a set V such that $V \in X$ and for every set Z such that $Z \in X$ holds $V \subseteq Z$.

Let us consider M, F. Then $\bigcup F$ is a subset of the carrier of M.

Let D be a non-empty set. Then Ω_D is a subset of D. Then $\emptyset_D$ is a subset of D.

Let us consider M. We say that M is totally bounded if and only if:

(Def.1) for every r such that $r > 0$ there exists F such that F is finite and the carrier of $M = \bigcup F$ and for every A such that $A \in F$ there exists g such that $A = \mathrm{Ball}(g, r)$.

Let us consider M. A function is called a sequence of M if:

(Def.2) $\mathrm{dom}\,\mathrm{it} = \mathbb{N}$ and $\mathrm{rng}\,\mathrm{it} \subseteq$ the carrier of M.

In the sequel S_1 will denote a sequence of M. The following proposition is true

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(5) f is a sequence of M if and only if $\operatorname{dom} f = \mathbb{N}$ and for every n holds $f(n)$ is an element of the carrier of M.

Let us consider M, S_1, n. Then $S_1(n)$ is an element of the carrier of M.

Let us consider M, S_1. We say that S_1 is convergent if and only if:

(Def.3) there exists an element x of the carrier of M such that for every r such that $r > 0$ there exists n such that for every m such that $n \leq m$ holds $\rho(S_1(m), x) < r$.

Let us consider M, S_1. Let us assume that S_1 is convergent. The functor $\lim S_1$ yields an element of the carrier of M and is defined by:

(Def.4) for every r such that $r > 0$ there exists n such that for every m such that $m \geq n$ holds $\rho(S_1(m), \lim S_1) < r$.

The following proposition is true

(6) For every S_1 such that S_1 is convergent holds $\lim S_1 = g$ if and only if for every r such that $0 < r$ there exists n such that for every m such that $n \leq m$ holds $\rho(S_1(m), g) < r$.

Let us consider M, S_1. We say that S_1 is a Cauchy sequence if and only if:

(Def.5) for every r such that $r > 0$ there exists p such that for all n, m such that $p \leq n$ and $p \leq m$ holds $\rho(S_1(n), S_1(m)) < r$.

Let us consider M. We say that M is complete if and only if:

(Def.6) for every S_1 such that S_1 is a Cauchy sequence holds S_1 is convergent.

We now state two propositions:

(7) For every S_1 such that S_1 is convergent holds S_1 is a Cauchy sequence.

(8) For every S_1 holds S_1 is a Cauchy sequence if and only if for every r such that $r > 0$ there exists p such that for all n, k such that $p \leq n$ holds $\rho(S_1(n+k), S_1(n)) < r$.

Let us consider M. A function from the carrier of M into the carrier of M is called a contraction of M if:

(Def.7) there exists L such that $0 < L$ and $L < 1$ and for all points x, y of M holds $\rho(\mathrm{it}(x), \mathrm{it}(y)) \leq L \cdot \rho(x, y)$.

We now state four propositions:

(9) For every contraction f of M such that M is complete there exists c such that $f(c) = c$ and for every element y of the carrier of M such that $f(y) = y$ holds $y = c$.

(10) If M_{top} is compact, then M is complete.

(11) For every contraction f of M such that M_{top} is compact there exists an element c of the carrier of M such that $f(c) = c$ and for every element y of the carrier of M such that $f(y) = y$ holds $y = c$.

(12) If M_{top} is compact, then M is totally bounded.

We now define two new predicates. Let us consider M. We say that M is bounded if and only if:

(Def.8) there exists r such that $0 < r$ and for all points x, y of M holds $\rho(x, y) \leq r$.

Let us consider A. We say that A is bounded if and only if:

(Def.9) (i) there exists r such that $0 < r$ and for all points x, y of M such that $x \in A$ and $y \in A$ holds $\rho(x, y) \leq r$ if $A \neq \emptyset$.

One can prove the following propositions:

(13) If $A \neq \emptyset$, then A is bounded if and only if there exists r such that $0 < r$ and for all points x, y of M such that $x \in A$ and $y \in A$ holds $\rho(x, y) \leq r$.

(14) $\emptyset_{\text{the carrier of } M}$ is bounded.

(15) If $A \neq \emptyset$, then A is bounded if and only if there exist r, c such that $0 < r$ and $c \in A$ and for every point z of M such that $z \in A$ holds $\rho(c, z) \leq r$.

(16) If $0 < r$, then $g \in \text{Ball}(g, r)$ and $\text{Ball}(g, r) \neq \emptyset$.

(17) If $r \leq 0$, then $\text{Ball}(g, r) = \emptyset$.

(18) If $0 < r$, then $\text{Ball}(g, r)$ is bounded.

(19) $\text{Ball}(g, r)$ is bounded.

(20) If A is bounded and B is bounded, then $A \cup B$ is bounded.

(21) If A is bounded and $B \subseteq A$, then B is bounded.

(22) If $A = \{g\}$, then A is bounded.

(23) If A is finite, then A is bounded.

(24) If F is finite and for every A such that $A \in F$ holds A is bounded, then $\bigcup F$ is bounded.

(25) M is bounded if and only if $\Omega_{\text{the carrier of } M}$ is bounded.

(26) If M is totally bounded, then M is bounded.

Let us consider M, A. Let us assume that $A \neq \emptyset$ and A is bounded. The functor $\vee A$ yields a real number and is defined as follows:

(Def.10) for all points x, y of M such that $x \in A$ and $y \in A$ holds $\rho(x, y) \leq \vee A$ and for every s such that for all points x, y of M such that $x \in A$ and $y \in A$ holds $\rho(x, y) \leq s$ holds $\vee A \leq s$.

We now state several propositions:

(27) Suppose $A \neq \emptyset$ and A is bounded. Then $\vee A = r$ if and only if for all points x, y of M such that $x \in A$ and $y \in A$ holds $\rho(x, y) \leq r$ and for every s such that for all points x, y of M such that $x \in A$ and $y \in A$ holds $\rho(x, y) \leq s$ holds $r \leq s$.

(28) If $A = \{g\}$, then $\vee A = 0$.

(29) If $A \neq \emptyset$ and A is bounded, then $0 \leq \vee A$.

(30) If $A \neq \emptyset$ and A is bounded, then $\vee A = 0$ if and only if there exists a point g of M such that $A = \{g\}$.

(31) If $0 < r$, then $\vee \text{Ball}(g, r) \leq 2 \cdot r$.

(32) If $A \neq \emptyset$ and A is bounded and $B \neq \emptyset$ and $B \subseteq A$, then B is bounded and $\vee B \leq \vee A$.

(33) If $A \neq \emptyset$ and A is bounded and $B \neq \emptyset$ and B is bounded and $A \cap B \neq \emptyset$, then $A \cup B$ is bounded and $\vee(A \cup B) \leq \vee A + \vee B$.

Let us consider M, S_1. Then $\operatorname{rng} S_1$ is a subset of the carrier of M.

One can prove the following proposition

(34) If S_1 is a Cauchy sequence, then $\operatorname{rng} S_1$ is bounded.

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