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Matrices. Abelian Group of Matrices

Katarzyna Jankowska
Warsaw University
Białystok

Summary. The basic conceptions of matrix algebra are introduced. The matrix is introduced as the finite sequence of sequences with the same length, i.e. as a sequence of lines. There are considered matrices over a field, and the fact that these matrices with addition form an Abelian group is proved.

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The notation and terminology used here have been introduced in the following papers: [9], [5], [6], [1], [8], [4], [2], [3], and [7]. For simplicity we adopt the following rules: x will be arbitrary, i, j, n, m will be natural numbers, D will be a non-empty set, K will be a field structure, s will be a finite sequence, a, a_1, a_2, b_1, b_2, d will be elements of D, p, p_1, p_2 will be finite sequences of elements of D, and F will be a field. A finite sequence is tabular if:

(Def.1) there exists a natural number n such that for every x such that $x \in \operatorname{rng}$ it there exists s such that $s = x$ and $\operatorname{len} s = n$.

The following propositions are true:

(1) $\langle\langle d\rangle\rangle$ is tabular.

(2) $m \longmapsto (n \longmapsto x)$ is tabular.

(3) For every s holds $\langle s\rangle$ is tabular.

(4) For all finite sequences s_1, s_2 such that $\operatorname{len} s_1 = n$ and $\operatorname{len} s_2 = n$ holds $\langle s_1, s_2\rangle$ is tabular.

(5) ε is tabular.

(6) $\langle\varepsilon, \varepsilon\rangle$ is tabular.

(7) $\langle\langle a_1\rangle, \langle a_2\rangle\rangle$ is tabular.

(8) $\langle\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\rangle$ is tabular.

A tabular finite sequence is non-trivial if:

(Def.2) there exists s such that $s \in \operatorname{rng}$ it and $\operatorname{len} s > 0$.

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Let D be a non-empty set.

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We now state the proposition

(9) s is a matrix over D if and only if there exists n such that for every x such that $x \in \operatorname{rng} s$ there exists p such that $x = p$ and $\operatorname{len} p = n$.

Let us consider D, m, n. A matrix over D is said to be a matrix over D of dimension $m \times n$ if:

(Def.3) $\operatorname{len} \mathrm{it} = m$ and for every p such that $p \in \operatorname{rng} \mathrm{it}$ holds $\operatorname{len} p = n$.

Let us consider D, n. A matrix over D of dimension n is a matrix over D of dimension $n \times n$.

We now define three new modes. Let us consider K. A matrix over K is a matrix over the carrier of K.

Let us consider n. A matrix over K of dimension n is a matrix over the carrier of K of dimension $n \times n$.

Let us consider m. A matrix over K of dimension $n \times m$ is a matrix over the carrier of K of dimension $n \times m$.

We now state a number of propositions:

(10) $m \longmapsto (n \longmapsto a)$ is a matrix over D of dimension $m \times n$.

(11) For every finite sequence p of elements of D holds $\langle p \rangle$ is a matrix over D of dimension $1 \times \operatorname{len} p$.

(12) For all p_1, p_2 such that $\operatorname{len} p_1 = n$ and $\operatorname{len} p_2 = n$ holds $\langle p_1, p_2 \rangle$ is a matrix over D of dimension $2 \times n$.

(13) ε is a matrix over D of dimension $0 \times m$.

(14) $\langle \varepsilon \rangle$ is a matrix over D of dimension 1×0.

(15) $\langle \langle a \rangle \rangle$ is a matrix over D of dimension 1.

(16) $\langle \varepsilon, \varepsilon \rangle$ is a matrix over D of dimension 2×0.

(17) $\langle \langle a_1, a_2 \rangle \rangle$ is a matrix over D of dimension 1×2.

(18) $\langle \langle a_1 \rangle, \langle a_2 \rangle \rangle$ is a matrix over D of dimension 2×1.

(19) $\langle \langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \rangle$ is a matrix over D of dimension 2.

In the sequel M, M_1, M_2 will be matrices over D. Let M be a tabular finite sequence. The functor $\operatorname{width} M$ yields a natural number and is defined as follows:

(Def.4) (i) there exists s such that $s \in \operatorname{rng} M$ and $\operatorname{len} s = \operatorname{width} M$ if $\operatorname{len} M > 0$,

(ii) $\operatorname{width} M = 0$, otherwise.

Next we state the proposition

(20) If $\operatorname{len} M > 0$, then for every n holds M is a matrix over D of dimension $\operatorname{len} M \times n$ if and only if $n = \operatorname{width} M$.

Let M be a tabular finite sequence. The indices of M yielding a set is defined by:

(Def.5) the indices of $M = [\!: \operatorname{Seg} \operatorname{len} M, \operatorname{Seg} \operatorname{width} M :\!]$.

Let us consider D, and let M be a matrix over D, and let us consider i, j. Let us assume that $\langle i, j \rangle \in$ the indices of M. The functor $M_{i,j}$ yielding an element of D is defined as follows:

(Def.6) there exists p such that $p = M(i)$ and $M_{i,j} = p(j)$.

The following proposition is true

(21) If $\operatorname{len} M_1 = \operatorname{len} M_2$ and $\operatorname{width} M_1 = \operatorname{width} M_2$ and for all i, j such that $\langle i, j \rangle \in$ the indices of M_1 holds $M_{1 i,j} = M_{2 i,j}$, then $M_1 = M_2$.

In this article we present several logical schemes. The scheme *MatrixLambda* deals with a non-empty set $\mathcal{A}$, a natural number $\mathcal{B}$, a natural number $\mathcal{C}$, and a binary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$ and states that:

there exists a matrix M over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{C}$ such that for all i, j such that $\langle i, j \rangle \in$ the indices of M holds $M_{i,j} = \mathcal{F}(i, j)$
for all values of the parameters.

The scheme *MatrixEx* concerns a non-empty set $\mathcal{A}$, a natural number $\mathcal{B}$, a natural number $\mathcal{C}$, and a ternary predicate $\mathcal{P}$, and states that:

there exists a matrix M over $\mathcal{A}$ of dimension $\mathcal{B} \times \mathcal{C}$ such that for all i, j such that $\langle i, j \rangle \in$ the indices of M holds $\mathcal{P}[i, j, M_{i,j}]$
provided the parameters have the following properties:

- for all i, j such that $\langle i, j \rangle \in [\!: \operatorname{Seg} \mathcal{B}, \operatorname{Seg} \mathcal{C} :\!]$ for all elements x_1, x_2 of $\mathcal{A}$ such that $\mathcal{P}[i, j, x_1]$ and $\mathcal{P}[i, j, x_2]$ holds $x_1 = x_2$,
- for all i, j such that $\langle i, j \rangle \in [\!: \operatorname{Seg} \mathcal{B}, \operatorname{Seg} \mathcal{C} :\!]$ there exists an element x of $\mathcal{A}$ such that $\mathcal{P}[i, j, x]$.

The scheme *SeqDLambda* concerns a non-empty set $\mathcal{A}$, a natural number $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$ and states that:

there exists a finite sequence p of elements of $\mathcal{A}$ such that $\operatorname{len} p = \mathcal{B}$ and for every i such that $i \in \operatorname{Seg} \mathcal{B}$ holds $p(i) = \mathcal{F}(i)$
for all values of the parameters.

We now state several propositions:

(22) For every matrix M over D of dimension $n \times m$ such that $\operatorname{len} M = 0$ holds $\operatorname{width} M = 0$.

(23) For every matrix M over D of dimension $0 \times m$ holds $\operatorname{len} M = 0$ and $\operatorname{width} M = 0$ and the indices of $M = \emptyset$.

(24) If $n > 0$, then for every matrix M over D of dimension $n \times m$ holds $\operatorname{len} M = n$ and $\operatorname{width} M = m$ and the indices of $M = [\!: \operatorname{Seg} n, \operatorname{Seg} m :\!]$.

(25) For every matrix M over D of dimension n holds $\operatorname{len} M = n$ and $\operatorname{width} M = n$ and the indices of $M = [\!: \operatorname{Seg} n, \operatorname{Seg} n :\!]$.

(26) For every matrix M over D of dimension $n \times m$ holds $\operatorname{len} M = n$ and the indices of $M = [\!: \operatorname{Seg} n, \operatorname{Seg} \operatorname{width} M :\!]$.

(27) For all matrices M_1, M_2 over D of dimension $n \times m$ holds the indices of M_1 = the indices of M_2.

(28) For all matrices M_1, M_2 over D of dimension $n \times m$ such that for all i, j such that $\langle i, j \rangle \in$ the indices of M_1 holds $M_{1 i,j} = M_{2 i,j}$ holds $M_1 = M_2$.

(29) For every matrix M_1 over D of dimension n and for all i, j such that $\langle i, j\rangle \in$ the indices of M_1 holds $\langle j, i\rangle \in$ the indices of M_1.

Let us consider D, and let M be a matrix over D. The functor M^{T} yielding a matrix over D is defined as follows:

(Def.7) $\mathrm{len}(M^{\mathrm{T}}) = \mathrm{width}\, M$ and for all i, j holds $\langle i, j\rangle \in$ the indices of M^{T} if and only if $\langle j, i\rangle \in$ the indices of M and for all i, j such that $\langle j, i\rangle \in$ the indices of M holds $M^{\mathrm{T}}_{i,j} = M_{j,i}$.

We now define two new functors. Let us consider D, M, i. The functor $\mathrm{Line}(M, i)$ yields a finite sequence of elements of D and is defined by:

(Def.8) $\mathrm{len}\,\mathrm{Line}(M, i) = \mathrm{width}\, M$ and for every j such that $j \in \mathrm{Seg}\,\mathrm{width}\, M$ holds $\mathrm{Line}(M, i)(j) = M_{i,j}$.

The functor $M_{\Box,i}$ yields a finite sequence of elements of D and is defined as follows:

(Def.9) $\mathrm{len}(M_{\Box,i}) = \mathrm{len}\, M$ and for every j such that $j \in \mathrm{Seg}\,\mathrm{len}\, M$ holds $M_{\Box,i}(j) = M_{j,i}$.

Let us consider D, and let M be a matrix over D, and let us consider i. Then $\mathrm{Line}(M, i)$ is an element of $D^{\mathrm{width}\, M}$. Then $M_{\Box,i}$ is an element of $D^{\mathrm{len}\, M}$.

In the sequel A, B are matrices over K of dimension n. We now define five new functors. Let us consider K, n. The functor $K^{n\times n}$ yields a non-empty set and is defined as follows:

(Def.10) $K^{n\times n} = (\,(\text{the carrier of } K)^n)^n$.

The functor $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n\times n}$ yielding a matrix over K of dimension n is defined as follows:

(Def.11) $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n\times n} = n \longmapsto (n \longmapsto 0_K)$.

The functor $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ yielding a matrix over K of dimension n is defined as follows:

(Def.12) for every i such that $\langle i, i\rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ holds $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})_{i,i} = 1_K$ and for all i, j such that $\langle i, j\rangle \in$ the indices of $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n}$ and $i \neq j$ holds $(\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}_K^{n\times n})_{i,j} = 0_K$.

Let us consider A. The functor $-A$ yielding a matrix over K of dimension n is defined as follows:

(Def.13) for all i, j such that $\langle i, j\rangle \in$ the indices of A holds $(-A)_{i,j} = -A_{i,j}$.

Let us consider B. The functor $A+B$ yielding a matrix over K of dimension n is defined by:

(Def.14) for all i, j such that $\langle i, j\rangle \in$ the indices of A holds $(A+B)_{i,j} = A_{i,j}+B_{i,j}$.

The following two propositions are true:

(30) For all i, j such that $\langle i, j\rangle \in$ the indices of $\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n\times n}$ holds $(\begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_K^{n\times n})_{i,j} = 0_K$.

(31) For every x holds x is an element of $K^{n\times n}$ if and only if x is a matrix over K of dimension n.

Let us consider K, n. A matrix over K of dimension n is called a diagonal n-dimensional matrix over K if:

(Def.15) for all i, j such that $\langle i, j\rangle \in$ the indices of it and $\mathrm{it}_{i,j} \neq 0_K$ holds $i = j$.

In the sequel A, B, C will denote matrices over F of dimension n. One can prove the following four propositions:

(32) $A+B = B+A$.

(33) $A+B+C = A+(B+C)$.

(34) $A + \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_F^{n\times n} = A$.

(35) $A + -A = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_F^{n\times n}$.

Let us consider F, n. The functor $F_{\mathrm{G}}^{n\times n}$ yielding an Abelian group is defined by:

(Def.16) the carrier of $F_{\mathrm{G}}^{n\times n} = F^{n\times n}$ and for all A, B holds (the addition of $F_{\mathrm{G}}^{n\times n})(A, B) = A+B$ and for every A holds (the reverse-map of $F_{\mathrm{G}}^{n\times n})(A) = -A$ and the zero of $F_{\mathrm{G}}^{n\times n} = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}_F^{n\times n}$.

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References

[1] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[3] Czesław Byliński. Binary operations applied to finite sequences. *Formalized Mathematics*, 1(**4**):643–649, 1990.
[4] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[5] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[6] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[7] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[8] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

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