FORMALIZED MATHEMATICS
Vol.2, No.4, September–October 1991
Université Catholique de Louvain

Introduction to Banach and Hilbert Spaces - Part III

Jan Popiołek
Warsaw University
Białystok

Summary. A continuation of [11] and of [12]. First we define the following concepts: the Cauchy sequence, the bounded sequence and the subsequence. The last part consists definitions of the complete space and the Hilbert space.

MML Identifier: BHSP_3.

The articles [5], [18], [22], [3], [4], [1], [10], [8], [9], [7], [15], [2], [23], [16], [17], [14], [21], [20], [19], [13], [11], [12], and [6] provide the notation and terminology for this paper. For simplicity we follow the rules: X is a real unitary space, x is a point of X, g is a point of X, a, r are real numbers, M is a real number, s_1, s_2, s_3, s_4 are sequences of X, N_1 is an increasing sequence of naturals, and k, n, m are natural numbers. Let us consider X, s_1. We say that s_1 is a Cauchy sequence if and only if:

(Def.1) for every r such that $r > 0$ there exists k such that for all n, m such that $n \geq k$ and $m \geq k$ holds $\rho(s_1(n), s_1(m)) < r$.

One can prove the following propositions:

(1) If s_1 is constant, then s_1 is a Cauchy sequence.

(2) s_1 is a Cauchy sequence if and only if for every r such that $r > 0$ there exists k such that for all n, m such that $n \geq k$ and $m \geq k$ holds $\|s_1(n) - s_1(m)\| < r$.

(3) If s_2 is a Cauchy sequence and s_3 is a Cauchy sequence, then $s_2 + s_3$ is a Cauchy sequence.

(4) If s_2 is a Cauchy sequence and s_3 is a Cauchy sequence, then $s_2 - s_3$ is a Cauchy sequence.

(5) If s_1 is a Cauchy sequence, then $a \cdot s_1$ is a Cauchy sequence.

(6) If s_1 is a Cauchy sequence, then $-s_1$ is a Cauchy sequence.

ISSN 0777–4028

(7) If s_1 is a Cauchy sequence, then $s_1 + x$ is a Cauchy sequence.

(8) If s_1 is a Cauchy sequence, then $s_1 - x$ is a Cauchy sequence.

(9) If s_1 is convergent, then s_1 is a Cauchy sequence.

Let us consider X, s_2, s_3. We say that s_2 is compared to s_3 if and only if:

(Def.2) for every r such that $r > 0$ there exists m such that for every n such that $n \geq m$ holds $\rho(s_2(n), s_3(n)) < r$.

One can prove the following propositions:

(10) s_1 is compared to s_1.

(11) If s_2 is compared to s_3, then s_3 is compared to s_2.

(12) If s_2 is compared to s_3 and s_3 is compared to s_4, then s_2 is compared to s_4.

(13) s_2 is compared to s_3 if and only if for every r such that $r > 0$ there exists m such that for every n such that $n \geq m$ holds $\|s_2(n) - s_3(n)\| < r$.

(14) If there exists k such that for every n such that $n \geq k$ holds $s_2(n) = s_3(n)$, then s_2 is compared to s_3.

(15) If s_2 is a Cauchy sequence and s_2 is compared to s_3, then s_3 is a Cauchy sequence.

(16) If s_2 is convergent and s_2 is compared to s_3, then s_3 is convergent.

(17) If s_2 is convergent and $\lim s_2 = g$ and s_2 is compared to s_3, then s_3 is convergent and $\lim s_3 = g$.

Let us consider X, s_1. We say that s_1 is bounded if and only if:

(Def.3) there exists M such that $M > 0$ and for every n holds $\|s_1(n)\| \leq M$.

One can prove the following propositions:

(18) If s_2 is bounded and s_3 is bounded, then $s_2 + s_3$ is bounded.

(19) If s_1 is bounded, then $-s_1$ is bounded.

(20) If s_2 is bounded and s_3 is bounded, then $s_2 - s_3$ is bounded.

(21) If s_1 is bounded, then $a \cdot s_1$ is bounded.

(22) If s_1 is constant, then s_1 is bounded.

(23) For every m there exists M such that $M > 0$ and for every n such that $n \leq m$ holds $\|s_1(n)\| < M$.

(24) If s_1 is convergent, then s_1 is bounded.

(25) If s_2 is bounded and s_2 is compared to s_3, then s_3 is bounded.

Let us consider X, N_1, s_1. Then $s_1 \cdot N_1$ is a sequence of X.

Let us consider X, s_2, s_1. We say that s_2 is a subsequence of s_1 if and only if:

(Def.4) there exists N_1 such that $s_2 = s_1 \cdot N_1$.

One can prove the following propositions:

(26) For every n holds $(s_1 \cdot N_1)(n) = s_1(N_1(n))$.

(27) s_1 is a subsequence of s_1.

(28) If s_2 is a subsequence of s_3 and s_3 is a subsequence of s_4, then s_2 is a subsequence of s_4.

(29) If s_1 is constant and s_2 is a subsequence of s_1, then s_2 is constant.

(30) If s_1 is constant and s_2 is a subsequence of s_1, then $s_1 = s_2$.

(31) If s_1 is bounded and s_2 is a subsequence of s_1, then s_2 is bounded.

(32) If s_1 is convergent and s_2 is a subsequence of s_1, then s_2 is convergent.

(33) If s_2 is a subsequence of s_1 and s_1 is convergent, then $\lim s_2 = \lim s_1$.

(34) If s_1 is a Cauchy sequence and s_2 is a subsequence of s_1, then s_2 is a Cauchy sequence.

Let us consider X, s_1, k. The functor $s_1 \uparrow k$ yields a sequence of X and is defined by:

(Def.5) for every n holds $(s_1 \uparrow k)(n) = s_1(n+k)$.

The following propositions are true:

(35) $s_1 \uparrow 0 = s_1$.

(36) $s_1 \uparrow k \uparrow m = s_1 \uparrow m \uparrow k$.

(37) $s_1 \uparrow k \uparrow m = s_1 \uparrow (k+m)$.

(38) $(s_2 + s_3) \uparrow k = s_2 \uparrow k + s_3 \uparrow k$.

(39) $(-s_1) \uparrow k = -s_1 \uparrow k$.

(40) $(s_2 - s_3) \uparrow k = s_2 \uparrow k - s_3 \uparrow k$.

(41) $(a \cdot s_1) \uparrow k = a \cdot (s_1 \uparrow k)$.

(42) $(s_1 \cdot N_1) \uparrow k = s_1 \cdot (N_1 \uparrow k)$.

(43) $s_1 \uparrow k$ is a subsequence of s_1.

(44) If s_1 is convergent, then $s_1 \uparrow k$ is convergent and $\lim(s_1 \uparrow k) = \lim s_1$.

(45) If s_1 is convergent and there exists k such that $s_2 = s_1 \uparrow k$, then s_2 is convergent and $\lim s_2 = \lim s_1$.

(46) If s_1 is convergent and there exists k such that $s_1 = s_2 \uparrow k$, then s_2 is convergent.

(47) If s_1 is a Cauchy sequence and there exists k such that $s_1 = s_2 \uparrow k$, then s_2 is a Cauchy sequence.

(48) If s_1 is a Cauchy sequence, then $s_1 \uparrow k$ is a Cauchy sequence.

(49) If s_2 is compared to s_3, then $s_2 \uparrow k$ is compared to $s_3 \uparrow k$.

(50) If s_1 is bounded, then $s_1 \uparrow k$ is bounded.

(51) If s_1 is constant, then $s_1 \uparrow k$ is constant.

Let us consider X. We say that X is a complete space if and only if:

(Def.6) for every s_1 such that s_1 is a Cauchy sequence holds s_1 is convergent.

The following propositions are true:

(52) If X is a complete space and s_2 is a Cauchy sequence and s_2 is compared to s_3, then s_3 is a Cauchy sequence.

(53) If X is a complete space and s_1 is a Cauchy sequence, then s_1 is bounded.

Let us consider X. We say that X is a Hilbert space if and only if:

(Def.7) X is a real unitary space and X is a complete space.

References

[1] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.

[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[6] Krzysztof Hryniewiecki. Recursive definitions. *Formalized Mathematics*, 1(**2**):321–328, 1990.

[7] Jarosław Kotowicz. Convergent real sequences. Upper and lower bound of sets of real numbers. *Formalized Mathematics*, 1(**3**):477–481, 1990.

[8] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[9] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(**3**):471–475, 1990.

[10] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[11] Jan Popiołek. Introduction to Banach and Hilbert spaces - part I. *Formalized Mathematics*, 2(**4**):511–516, 1991.

[12] Jan Popiołek. Introduction to Banach and Hilbert spaces - part II. *Formalized Mathematics*, 2(**4**):517–521, 1991.

[13] Jan Popiołek. Quadratic inequalities. *Formalized Mathematics*, 2(**4**):507–509, 1991.

[14] Jan Popiołek. Some properties of functions modul and signum. *Formalized Mathematics*, 1(**2**):263–264, 1990.

[15] Konrad Raczkowski and Andrzej Nędzusiak. Serieses. *Formalized Mathematics*, 2(**4**):449–452, 1991.

[16] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[17] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[18] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[19] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

[20] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Formalized Mathematics*, 1(**2**):297–301, 1990.

[21] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.

[22] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1(**1**):17–23, 1990.

[23] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

Received July 19, 1991