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Algebra of Normal Forms Is a Heyting Algebra [1]

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Summary. We prove that the lattice of normal forms over an arbitrary set, introduced in [7], is an implicative lattice. The relative psedo-complement $\alpha \Rightarrow \beta$ is defined as $\bigsqcup_{\alpha_1 \cup \alpha_2 = \alpha} -\alpha_1 \sqcap \alpha_2 \rightarrowtail \beta$, where $-\alpha$ is the pseudo-complement of α and $\alpha \rightarrowtail \beta$ is a rather strong implication introduced in this paper.

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The articles [13], [4], [5], [2], [14], [3], [8], [6], [15], [9], [16], [10], [11], [12], [7], and [1] provide the notation and terminology for this paper. One can prove the following proposition

(1) For all non-empty sets A, B, C and for every function f from A into B such that for every element x of A holds $f(x) \in C$ holds f is a function from A into C.

In the sequel A will be a non-empty set and a will be an element of A. Let us consider A, and let B, C be elements of $\mathrm{Fin}\, A$. Let us note that one can characterize the predicate $B \subseteq C$ by the following (equivalent) condition:

(Def.1) for every a such that $a \in B$ holds $a \in C$.

Let A be a non-empty set, and let B be a non-empty subset of A. Then $\overset{B}{\hookrightarrow}$ is a function from B into A.

The following proposition is true

(2) For every non-empty set A and for every non-empty subset B of A and for every element x of B holds $(\overset{B}{\hookrightarrow})(x) = x$.

In the sequel A denotes a set. Let us consider A. Let us assume that A is non-empty. The functor $[A]$ yielding an non-empty set is defined by:

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(Def.2) $[A] = A$.

We follow the rules: B, C will denote elements of $\mathrm{Fin}\,\mathrm{DP}(A)$, a, b, c, s, t_1, t_2 will denote elements of $\mathrm{DP}(A)$, and u, v, w will denote elements of the carrier of the lattice of normal forms over A. The following propositions are true:

(3) If $B = \emptyset$, then $\mu B = \emptyset$.

(4) For an arbitrary x such that $x \in B$ holds x is an element of $\mathrm{DP}(A)$.

Let us consider A, a. Then $\{a\}$ is an element of the normal forms over A.

Let us consider A, and let u be an element of the carrier of the lattice of normal forms over A.

The functor ${}^{@}u$ yields an element of the normal forms over A and is defined as follows:

(Def.3) ${}^{@}u = u$.

One can prove the following two propositions:

(5) $\sqcap_A({}^{@}u, {}^{@}v) =$ (the meet operation of the lattice of normal forms over $A)(u, v)$.

(6) $\sqcup_A({}^{@}u, {}^{@}v) =$ (the join operation of the lattice of normal forms over $A)(u, v)$.

In the sequel K, L will denote elements of the normal forms over A. One can prove the following propositions:

(7) $\mu(K \frown K) = K$.

(8) For every set X such that $X \subseteq K$ holds $X \in$ the normal forms over A.

(9) $\emptyset$ is an element of the normal forms over A.

(10) For every set X such that $X \subseteq u$ holds X is an element of the carrier of the lattice of normal forms over A.

Let us consider A. The functor $\{\square\}_A$ yields a function from $\mathrm{DP}(A)$ into the carrier of the lattice of normal forms over A and is defined by:

(Def.4) $\{\square\}_A(a) = \{a\}$.

The following propositions are true:

(11) If $c \in \{\square\}_A(a)$, then $c = a$.

(12) $a \in \{\square\}_A(a)$.

(13) $\{\square\}_A(a) = \mathrm{singleton}_{\mathrm{DP}(A)}(a)$.

(14) $\bigsqcup^{\mathrm{f}}_{K}(\{\square\}_A) = \mathrm{FinUnion}(K, \mathrm{singleton}_{\mathrm{DP}(A)})$.

(15) $u = \bigsqcup^{\mathrm{f}}_{({}^{@}u)}(\{\square\}_A)$.

In the sequel f will denote an element of $[\!:\mathrm{Fin}\,A, \mathrm{Fin}\,A\,:\!]^{\mathrm{DP}(A)}$ and g will denote an element of $[A]^{\mathrm{DP}(A)}$. Let A be a set. The functor $\square \setminus_A \square$ yielding a binary operation on $[\!:\mathrm{Fin}\,A, \mathrm{Fin}\,A\,:\!]$ is defined as follows:

(Def.5) for all elements a, b of $[\!:\mathrm{Fin}\,A, \mathrm{Fin}\,A\,:\!]$ holds $\square \setminus_A \square(a, b) = a \setminus b$.

We now define two new functors. Let us consider A, B. The functor $-B$ yielding an element of $\mathrm{Fin}\,\mathrm{DP}(A)$ is defined by:

(Def.6) $-B = \mathrm{DP}(A) \cap \{\langle\{g(t_1) : g(t_1) \in t_{\mathbf{12}} \wedge t_1 \in B\}, \{g(t_2) : g(t_2) \in t_{\mathbf{21}} \wedge t_2 \in B\}\rangle : s \in B \Rightarrow g(s) \in s_{\mathbf{1}} \cup s_{\mathbf{2}}\}$.

Let us consider C. The functor $B \rightarrowtail C$ yielding an element of Fin DP(A) is defined by:

(Def.7) $B \rightarrowtail C = \mathrm{DP}(A) \cap \Big\{\mathrm{FinUnion}(B, \Box \setminus_A \Box^{\circ}(f, \overset{\mathrm{DP}(A)}{\hookrightarrow})) : f^{\circ} B \subseteq C\Big\}$.

The following propositions are true:

(16) Suppose $c \in -B$. Then there exists g such that for every s such that $s \in B$ holds $g(s) \in s_{\mathbf{1}} \cup s_{\mathbf{2}}$ and $c = \langle\, \{g(t_1) : g(t_1) \in t_{\mathbf{12}} \wedge t_1 \in B\}\,, \{g(t_2) : g(t_2) \in t_{\mathbf{21}} \wedge t_2 \in B\}\,\rangle$.

(17) $\langle\emptyset, \emptyset\rangle$ is an element of DP(A).

(18) For every K such that $K = \emptyset$ holds $-K = \{\langle\emptyset, \emptyset\rangle\}$.

(19) For all K, L such that $K = \emptyset$ and $L = \emptyset$ holds $K \rightarrowtail L = \{\langle\emptyset, \emptyset\rangle\}$.

(20) For every element a of DP$(\emptyset)$ holds $a = \langle\emptyset, \emptyset\rangle$.

(21) $\mathrm{DP}(\emptyset) = \{\langle\emptyset, \emptyset\rangle\}$.

(22) $\{\langle\emptyset, \emptyset\rangle\}$ is an element of the normal forms over A.

(23) If $c \in B \rightarrowtail C$, then there exists f such that $f^{\circ} B \subseteq C$ and $c = \mathrm{FinUnion}(B, \Box \setminus_A \Box^{\circ}(f, \overset{\mathrm{DP}(A)}{\hookrightarrow}))$.

(24) If $K \frown \{a\} = \emptyset$, then there exists b such that $b \in -K$ and $b \subseteq a$.

(25) If for every b such that $b \in u$ holds $b \cup a \in \mathrm{DP}(A)$ and for every c such that $c \in u$ there exists b such that $b \in v$ and $b \subseteq c \cup a$, then there exists b such that $b \in ({}^{@}u) \rightarrowtail {}^{@}v$ and $b \subseteq a$.

(26) $K \frown -K = \emptyset$.

We now define four new functors. Let us consider A. The functor $\Box^{\mathrm{c}}{}_A$ yielding a unary operation on the carrier of the lattice of normal forms over A is defined by:

(Def.8) $\Box^{\mathrm{c}}{}_A(u) = \mu(-{}^{@}u)$.

The functor $\Box \rightarrowtail_A \Box$ yields a binary operation on the carrier of the lattice of normal forms over A and is defined by:

(Def.9) $(\Box \rightarrowtail_A \Box)(u, v) = \mu(({}^{@}u) \rightarrowtail {}^{@}v)$.

Let us consider u. The functor 2^u yielding an element of Fin (the carrier of the lattice of normal forms over A) is defined by:

(Def.10) $2^u = 2^u$.

The functor $\Box \setminus_u \Box$ yielding a unary operation on the carrier of the lattice of normal forms over A is defined as follows:

(Def.11) $(\Box \setminus_u \Box)(v) = u \setminus v$.

We now state several propositions:

(27) $(\Box \setminus_u \Box)(v) \sqsubseteq u$.

(28) $u \sqcap \Box^{\mathrm{c}}{}_A(u) = \bot_{\text{the lattice of normal forms over } A}$.

(29) $u \sqcap (\Box \rightarrowtail_A \Box)(u,\, v) \sqsubseteq v$.

(30) If $({}^{@}u) \frown \{a\} = \emptyset$, then $\{\Box\}_A(a) \sqsubseteq \Box^{c}{}_A(u)$.

(31) If for every b such that $b \in u$ holds $b \cup a \in \mathrm{DP}(A)$ and $u \sqcap \{\Box\}_A(a) \sqsubseteq w$, then $\{\Box\}_A(a) \sqsubseteq (\Box \rightarrowtail_A \Box)(u,\, w)$.

(32) The lattice of normal forms over A is an implicative lattice.

(33) $u \Rightarrow v = \bigsqcup^{\mathrm{f}}_{2^u}($ (the meet operation of the lattice of normal forms over $A)^{\circ}(\Box^{c}{}_A,\, (\Box \rightarrowtail_A \Box)^{\circ}(\Box \setminus_u \Box, v)))$.

(34) $\top_{\text{The lattice of normal forms over } A} = \{\langle \emptyset, \emptyset \rangle\}$.

References

[1] Grzegorz Bancerek. Filters - part I. *Formalized Mathematics*, 1(**5**):813–819, 1990.
[2] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(**1**):245–254, 1990.
[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[6] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[7] Andrzej Trybulec. Algebra of normal forms. *Formalized Mathematics*, 2(**2**):237–242, 1991.
[8] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[9] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.
[10] Andrzej Trybulec. Finite join and finite meet and dual lattices. *Formalized Mathematics*, 1(**5**):983–988, 1990.
[11] Andrzej Trybulec. Function domains and Frænkel operator. *Formalized Mathematics*, 1(**3**):495–500, 1990.
[12] Andrzej Trybulec. Semilattice operations on finite subsets. *Formalized Mathematics*, 1(**2**):369–376, 1990.
[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.
[14] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.
[15] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1(**1**):187–190, 1990.
[16] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(**1**):215–222, 1990.

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