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Filters - Part II. Quotient Lattices Modulo Filters and Direct Product of Two Lattices

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Summary. Binary and unary operation preserving binary relations and quotients of those operations modulo equivalence relations are introduced. It is shown that the quotients inherit some important properties (commutativity, associativity, distributivity, ect.). Based on it the quotient (also called factor) lattice modulo filter (ie. modulo the equivalence relation w.r.t the filter) is introduced. Similarly, some properties of the direct product of two binary (unary) operations are presented and then the direct product of two lattices is introduced. Besides, the heredity of distributivity, modularity, completeness, etc., for the product of lattices is also shown. Finally, the concept of isomorphic lattices is introduced, and it is shown that every Boolean lattice B is isomorphic with the direct product of the factor lattice $B/[a]$ and the lattice latt$[a]$, where a is an element of B.

MML Identifier: `FILTER_1`.

The notation and terminology used in this paper are introduced in the following papers: [11], [5], [6], [13], [4], [8], [12], [9], [2], [3], [7], [14], [1], and [10]. Let L be a lattice structure. An element of L is an element of the carrier of L.

For simplicity we adopt the following convention: L, L_1, L_2 denote lattices, F_1, F_2 denote filters of L, p, q denote elements of L, p_1, q_1 denote elements of L_1, p_2, q_2 denote elements of L_2, x, x_1, y, y_1 are arbitrary, D, D_1, D_2 denote non-empty sets, R denotes a binary relation, R_1 denotes an equivalence relation of D, a, b, d denote elements of D, a_1, b_1 denote elements of D_1, a_2, b_2 denote elements of D_2, B denotes a boolean lattice, F_3 denotes a filter of B, I denotes an implicative lattice, F_4 denotes a filter of I, i, i_1, i_2, j, j_1, j_2, k denote elements of I, f_1, g_1 denote binary operations on D_1, and f_2, g_2 denote binary operations on D_2. One can prove the following two propositions:

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(1) $F_1 \cap F_2$ is a filter of L.

(2) If $[p] = [q]$, then $p = q$.

Let us consider L, F_1, F_2. Then $F_1 \cap F_2$ is a filter of L.

We now define two new modes. Let us consider D, R. A unary operation on D is called a unary R-congruent operation on D if:

(Def.1) for all elements x, y of D such that $\langle x, y\rangle \in R$ holds $\langle \mathrm{it}(x), \mathrm{it}(y)\rangle \in R$.

A binary operation on D is called a binary R-congruent operation on D if:

(Def.2) for all elements x_1, y_1, x_2, y_2 of D such that $\langle x_1, y_1\rangle \in R$ and $\langle x_2, y_2\rangle \in R$ holds $\langle \mathrm{it}(x_1, x_2), \mathrm{it}(y_1, y_2)\rangle \in R$.

In the sequel F, G denote binary R_1-congruent operations on D. We now define two new modes. Let us consider D, and let R be an equivalence relation of D. A unary operation on R is a unary R-congruent operation on D.

A binary operation on R is a binary R-congruent operation on D.

Then $\mathrm{Classes}\, R$ is an non-empty subset of 2^D.

Let X be a set, and let S be a non-empty subset of 2^X. We see that the element of S is a subset of X.

Let us consider D, and let R be an equivalence relation of D, and let d be an element of D. Then $[d]_R$ is an element of $\mathrm{Classes}\, R$.

Let us consider D, and let R be an equivalence relation of D, and let u be a unary operation on D. Let us assume that u is a unary R-congruent operation on D. The functor $u_{/R}$ yielding a unary operation on $\mathrm{Classes}\, R$ is defined as follows:

(Def.3) for all x, y such that $x \in \mathrm{Classes}\, R$ and $y \in x$ holds $u_{/R}(x) = [u(y)]_R$.

Let us consider D, and let R be an equivalence relation of D, and let b be a binary operation on D. Let us assume that b is a binary R-congruent operation on D. The functor $b_{/R}$ yields a binary operation on $\mathrm{Classes}\, R$ and is defined by:

(Def.4) for all x, y, x_1, y_1 such that $x \in \mathrm{Classes}\, R$ and $y \in \mathrm{Classes}\, R$ and $x_1 \in x$ and $y_1 \in y$ holds $b_{/R}(x, y) = [b(x_1, y_1)]_R$.

We now state the proposition

(3) $F_{/R_1}([a]_{R_1}, [b]_{R_1}) = [F(a, b)]_{R_1}$.

The following propositions are true:

(4) If F is commutative, then $F_{/R_1}$ is commutative.

(5) If F is associative, then $F_{/R_1}$ is associative.

(6) If d is a left unity w.r.t. F, then $[d]_{R_1}$ is a left unity w.r.t. $F_{/R_1}$.

(7) If d is a right unity w.r.t. F, then $[d]_{R_1}$ is a right unity w.r.t. $F_{/R_1}$.

(8) If d is a unity w.r.t. F, then $[d]_{R_1}$ is a unity w.r.t. $F_{/R_1}$.

(9) If F is left distributive w.r.t. G, then $F_{/R_1}$ is left distributive w.r.t. $G_{/R_1}$.

(10) If F is right distributive w.r.t. G, then $F_{/R_1}$ is right distributive w.r.t. $G_{/R_1}$.

(11) If F is distributive w.r.t. G, then $F_{/R_1}$ is distributive w.r.t. $G_{/R_1}$.

(12) If F absorbs G, then $F_{/R_1}$ absorbs $G_{/R_1}$.

(13) The join operation of I is a binary $\equiv_{F_4}$-congruent operation on the carrier of I.

(14) The meet operation of I is a binary $\equiv_{F_4}$-congruent operation on the carrier of I.

Let L be a lattice, and let F be a filter of L. Let us assume that L is an implicative lattice. The functor $L_{/F}$ yields a lattice and is defined as follows:

(Def.5) for every equivalence relation R of the carrier of L such that $R = \equiv_F$ holds $L_{/F} = \langle \mathrm{Classes}\, R$, (the join operation of $L)_{/R}$, (the meet operation of $L)_{/R}\rangle$.

Let L be a lattice, and let F be a filter of L, and let a be an element of L. Let us assume that L is an implicative lattice. The functor $a_{/F}$ yielding an element of $L_{/F}$ is defined as follows:

(Def.6) for every equivalence relation R of the carrier of L such that $R = \equiv_F$ holds $a_{/F} = [a]_R$.

Next we state several propositions:

(15) $i_{/F_4} \sqcup j_{/F_4} = (i \sqcup j)_{/F_4}$ and $i_{/F_4} \sqcap j_{/F_4} = (i \sqcap j)_{/F_4}$.

(16) $i_{/F_4} \sqsubseteq j_{/F_4}$ if and only if $i \Rightarrow j \in F_4$.

(17) $i \sqcap j \Rightarrow k = i \Rightarrow (j \Rightarrow k)$.

(18) If I is a lower bound lattice, then $I_{/F_4}$ is a lower bound lattice and $\bot_{I_{/F_4}} = (\bot_I)_{/F_4}$.

(19) $I_{/F_4}$ is an upper bound lattice and $\top_{I_{/F_4}} = (\top_I)_{/F_4}$.

(20) $I_{/F_4}$ is an implicative lattice.

(21) $B_{/F_3}$ is a boolean lattice.

Let D_1, D_2 be non-empty sets, and let f_1 be a binary operation on D_1, and let f_2 be a binary operation on D_2. Then $|{:}f_1, f_2{:}|$ is a binary operation on $[\!:D_1, D_2:\!]$.

We now state the proposition

(22) $|{:}f_1, f_2{:}|(\langle a_1, a_2\rangle, \langle b_1, b_2\rangle) = \langle f_1(a_1, b_1), f_2(a_2, b_2)\rangle$.

One can prove the following propositions:

(23) f_1 is commutative and f_2 is commutative if and only if $|{:}f_1, f_2{:}|$ is commutative.

(24) f_1 is associative and f_2 is associative if and only if $|{:}f_1, f_2{:}|$ is associative.

(25) a_1 is a left unity w.r.t. f_1 and a_2 is a left unity w.r.t. f_2 if and only if $\langle a_1, a_2\rangle$ is a left unity w.r.t. $|{:}f_1, f_2{:}|$.

(26) a_1 is a right unity w.r.t. f_1 and a_2 is a right unity w.r.t. f_2 if and only if $\langle a_1, a_2\rangle$ is a right unity w.r.t. $|{:}f_1, f_2{:}|$.

(27) a_1 is a unity w.r.t. f_1 and a_2 is a unity w.r.t. f_2 if and only if $\langle a_1, a_2\rangle$ is a unity w.r.t. $|{:}f_1, f_2{:}|$.

(28) f_1 is left distributive w.r.t. g_1 and f_2 is left distributive w.r.t. g_2 if and only if $|{:}f_1, f_2{:}|$ is left distributive w.r.t. $|{:}g_1, g_2{:}|$.

(29) f_1 is right distributive w.r.t. g_1 and f_2 is right distributive w.r.t. g_2 if and only if $|{:}f_1, f_2{:}|$ is right distributive w.r.t. $|{:}g_1, g_2{:}|$.

(30) f_1 is distributive w.r.t. g_1 and f_2 is distributive w.r.t. g_2 if and only if $|{:}f_1, f_2{:}|$ is distributive w.r.t. $|{:}g_1, g_2{:}|$.

(31) f_1 absorbs g_1 and f_2 absorbs g_2 if and only if $|{:}f_1, f_2{:}|$ absorbs $|{:}g_1, g_2{:}|$.

Let L_1, L_2 be lattice structures. The functor $[{:}\, L_1, L_2 \,{:}]$ yielding a lattice structure is defined by:

(Def.7) $[{:}\, L_1, L_2 \,{:}] = \langle [{:}$ the carrier of L_1, the carrier of $L_2 \,{:}], |{:}$ the join operation of L_1, the join operation of $L_2{:}|, |{:}$ the meet operation of L_1, the meet operation of $L_2{:}|\rangle$.

Let L be a lattice. The functor $\mathrm{LattRel}(L)$ yields a binary relation and is defined as follows:

(Def.8) $\mathrm{LattRel}(L) = \{\langle p, q\rangle : p \sqsubseteq q\}$, where p ranges over elements of the carrier of L, and q ranges over elements of the carrier of L.

We now state two propositions:

(32) $\langle p, q\rangle \in \mathrm{LattRel}(L)$ if and only if $p \sqsubseteq q$.

(33) $\mathrm{dom}\,\mathrm{LattRel}(L) =$ the carrier of L and $\mathrm{rng}\,\mathrm{LattRel}(L) =$ the carrier of L and $\mathrm{field}\,\mathrm{LattRel}(L) =$ the carrier of L.

Let L_1, L_2 be lattices. We say that L_1 and L_2 are isomorphic if and only if:

(Def.9) $\mathrm{LattRel}(L_1)$ and $\mathrm{LattRel}(L_2)$ are isomorphic.

Let us notice that the predicate introduced above is reflexive and symmetric. Then $[{:}\, L_1, L_2 \,{:}]$ is a lattice.

Next we state two propositions:

(34) For all lattices L_1, L_2, L_3 such that L_1 and L_2 are isomorphic and L_2 and L_3 are isomorphic holds L_1 and L_3 are isomorphic.

(35) For all L_1, L_2 being lattice structures such that $[{:}\, L_1, L_2 \,{:}]$ is a lattice holds L_1 is a lattice and L_2 is a lattice.

Let L_1, L_2 be lattices, and let a be an element of L_1, and let b be an element of L_2. Then $\langle a, b\rangle$ is an element of $[{:}\, L_1, L_2 \,{:}]$.

The following propositions are true:

(36) $\langle p_1, p_2\rangle \sqcup \langle q_1, q_2\rangle = \langle p_1 \sqcup q_1, p_2 \sqcup q_2\rangle$ and $\langle p_1, p_2\rangle \sqcap \langle q_1, q_2\rangle = \langle p_1 \sqcap q_1, p_2 \sqcap q_2\rangle$.

(37) $\langle p_1, p_2\rangle \sqsubseteq \langle q_1, q_2\rangle$ if and only if $p_1 \sqsubseteq q_1$ and $p_2 \sqsubseteq q_2$.

(38) L_1 is a modular lattice and L_2 is a modular lattice if and only if $[{:}\, L_1, L_2 \,{:}]$ is a modular lattice.

(39) L_1 is a distributive lattice and L_2 is a distributive lattice if and only if $[{:}\, L_1, L_2 \,{:}]$ is a distributive lattice.

(40) L_1 is a lower bound lattice and L_2 is a lower bound lattice if and only if $[{:}\, L_1, L_2 \,{:}]$ is a lower bound lattice.

(41) L_1 is an upper bound lattice and L_2 is an upper bound lattice if and only if $[\!: L_1, L_2 :\!]$ is an upper bound lattice.

(42) L_1 is a bound lattice and L_2 is a bound lattice if and only if $[\!: L_1, L_2 :\!]$ is a bound lattice.

(43) If L_1 is a lower bound lattice and L_2 is a lower bound lattice, then $\bot_{[\!: L_1, L_2 :\!]} = \langle \bot_{L_1}, \bot_{L_2} \rangle$.

(44) If L_1 is an upper bound lattice and L_2 is an upper bound lattice, then $\top_{[\!: L_1, L_2 :\!]} = \langle \top_{L_1}, \top_{L_2} \rangle$.

(45) If L_1 is a bound lattice and L_2 is a bound lattice, then p_1 is a complement of q_1 and p_2 is a complement of q_2 if and only if $\langle p_1, p_2 \rangle$ is a complement of $\langle q_1, q_2 \rangle$.

(46) L_1 is a complemented lattice and L_2 is a complemented lattice if and only if $[\!: L_1, L_2 :\!]$ is a complemented lattice.

(47) L_1 is a boolean lattice and L_2 is a boolean lattice if and only if $[\!: L_1, L_2 :\!]$ is a boolean lattice.

(48) L_1 is an implicative lattice and L_2 is an implicative lattice if and only if $[\!: L_1, L_2 :\!]$ is an implicative lattice.

(49) $[\!: L_1, L_2 :\!]^\circ = [\!: L_1{}^\circ, L_2{}^\circ :\!]$.

(50) $[\!: L_1, L_2 :\!]$ and $[\!: L_2, L_1 :\!]$ are isomorphic.

We follow the rules: B will be a boolean lattice and a, b, c, d will be elements of B. One can prove the following propositions:

(51) $a \Leftrightarrow b = a \sqcap b \sqcup a^{\rm c} \sqcap b^{\rm c}$.

(52) $(a \Rightarrow b)^{\rm c} = a \sqcap b^{\rm c}$ and $(a \Leftrightarrow b)^{\rm c} = a \sqcap b^{\rm c} \sqcup a^{\rm c} \sqcap b$ and $(a \Leftrightarrow b)^{\rm c} = a \Leftrightarrow b^{\rm c}$ and $(a \Leftrightarrow b)^{\rm c} = a^{\rm c} \Leftrightarrow b$.

(53) If $a \Leftrightarrow b = a \Leftrightarrow c$, then $b = c$.

(54) $a \Leftrightarrow (a \Leftrightarrow b) = b$.

(55) $i \sqcup j \Rightarrow i = j \Rightarrow i$ and $i \Rightarrow i \sqcap j = i \Rightarrow j$.

(56) $i \Rightarrow j \sqsubseteq i \Rightarrow j \sqcup k$ and $i \Rightarrow j \sqsubseteq i \sqcap k \Rightarrow j$ and $i \Rightarrow j \sqsubseteq i \Rightarrow k \sqcup j$ and $i \Rightarrow j \sqsubseteq k \sqcap i \Rightarrow j$.

(57) $(i \Rightarrow k) \sqcap (j \Rightarrow k) \sqsubseteq i \sqcup j \Rightarrow k$.

(58) $(i \Rightarrow j) \sqcap (i \Rightarrow k) \sqsubseteq i \Rightarrow j \sqcap k$.

(59) If $i_1 \Leftrightarrow i_2 \in F_4$ and $j_1 \Leftrightarrow j_2 \in F_4$, then $i_1 \sqcup j_1 \Leftrightarrow i_2 \sqcup j_2 \in F_4$ and $i_1 \sqcap j_1 \Leftrightarrow i_2 \sqcap j_2 \in F_4$.

(60) If $i \in [k]_{\equiv_{F_4}}$ and $j \in [k]_{\equiv_{F_4}}$, then $i \sqcup j \in [k]_{\equiv_{F_4}}$ and $i \sqcap j \in [k]_{\equiv_{F_4}}$.

(61) $c \sqcup (c \Leftrightarrow d) \in [c]_{\equiv_{[d]}}$ and for every b such that $b \in [c]_{\equiv_{[d]}}$ holds $b \sqsubseteq c \sqcup (c \Leftrightarrow d)$.

(62) B and $[\!: B_{/[a]}, \mathbb{L}_{[a]} :\!]$ are isomorphic.

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