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A Projective Closure and Projective Horizon of an Affine Space

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Summary. With every affine space A we correlate two incidence structures. The first, called Inc-ProjSp(A), is the usual projective closure of A, i.e. the structure obtained from A by adding directions of lines and planes of A. The second, called projective horizon of A, is the structure build from directions. We prove that Inc-ProjSp(A) is always a projective space, and projective horizon of A is a projective space provided A is at least 3-dimensional. Some evident relationships between projective and affine configurational axioms that may hold in A and in Inc-ProjSp(A) are established.

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The notation and terminology used in this paper have been introduced in the following articles: [9], [11], [12], [8], [6], [13], [10], [3], [4], [5], [1], [7], and [2]. We adopt the following rules: A_1 will denote an affine space, A, K, M, X, Y will denote subsets of the points of A_1, and x, y will be arbitrary. Next we state several propositions:

(1) If A_1 is an affine plane and $X =$ the points of A_1, then X is a plane.

(2) If A_1 is an affine plane and X is a plane, then $X =$ the points of A_1.

(3) If A_1 is an affine plane and X is a plane and Y is a plane, then $X = Y$.

(4) If $X =$ the points of A_1 and X is a plane, then A_1 is an affine plane.

(5) If $A \nparallel K$ and $A||X$ and $A||Y$ and $K||X$ and $K||Y$ and A is a line and K is a line and X is a plane and Y is a plane, then $X||Y$.

(6) If A is a line and X is a plane and Y is a plane and $A||X$ and $X||Y$, then $A||Y$.

Let D be a non-empty set, and let X be a set. Then $D \cup X$ is a non-empty set.

Let us consider A_1. The lines of A_1 yields a family of subsets of the points of A_1 and is defined as follows:

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(Def.1) the lines of $A_1 = \{A : A \text{ is a line }\}$.

Let us consider A_1. The planes of A_1 yielding a family of subsets of the points of A_1 is defined as follows:

(Def.2) the planes of $A_1 = \{A : A \text{ is a plane }\}$.

The following two propositions are true:

(7) For every x holds $x \in$ the lines of A_1 if and only if there exists X such that $x = X$ and X is a line.

(8) For every x holds $x \in$ the planes of A_1 if and only if there exists X such that $x = X$ and X is a plane.

Let us consider A_1. The parallelity of lines of A_1 yields an equivalence relation of the lines of A_1 and is defined by:

(Def.3) the parallelity of lines of $A_1 = \{\langle K, M\rangle : K \text{ is a line} \wedge M \text{ is a line} \wedge K || M\}$.

Let us consider A_1. The parallelity of planes of A_1 yielding an equivalence relation of the planes of A_1 is defined as follows:

(Def.4) the parallelity of planes of $A_1 = \{\langle X, Y\rangle : X \text{ is a plane} \wedge Y \text{ is a plane} \wedge X || Y\}$.

Let us consider A_1, X. Let us assume that X is a line. The direction of X yields a subset of the lines of A_1 and is defined by:

(Def.5) the direction of $X = [X]_{\text{the parallelity of lines of } A_1}$.

Let us consider A_1, X. Let us assume that X is a plane. The direction of X yielding a subset of the planes of A_1 is defined as follows:

(Def.6) the direction of $X = [X]_{\text{the parallelity of planes of } A_1}$.

Next we state several propositions:

(9) If X is a line, then for every x holds $x \in$ the direction of X if and only if there exists Y such that $x = Y$ and Y is a line and $X || Y$.

(10) If X is a plane, then for every x holds $x \in$ the direction of X if and only if there exists Y such that $x = Y$ and Y is a plane and $X || Y$.

(11) If X is a line and Y is a line, then the direction of $X =$ the direction of Y if and only if $X \parallel Y$.

(12) If X is a line and Y is a line, then the direction of $X =$ the direction of Y if and only if $X || Y$.

(13) If X is a plane and Y is a plane, then
the direction of $X =$ the direction of Y
if and only if $X || Y$.

Let us consider A_1. The directions of lines of A_1 yields a non-empty set and is defined as follows:

(Def.7) the directions of lines of $A_1 =$ Classes(the parallelity of lines of A_1).

Let us consider A_1. The directions of planes of A_1 yielding a non-empty set is defined by:

(Def.8) the directions of planes of A_1 = Classes(the parallelity of planes of A_1).

One can prove the following propositions:

(14) For every x holds $x \in$ the directions of lines of A_1 if and only if there exists X such that x = the direction of X and X is a line.

(15) For every x holds $x \in$ the directions of planes of A_1 if and only if there exists X such that x = the direction of X and X is a plane.

(16) (the points of A_1) $\cap$ the directions of lines of $A_1 = \emptyset$.

(17) If A_1 is an affine plane, then (the lines of A_1) $\cap$ the directions of planes of $A_1 = \emptyset$.

(18) For every x holds $x \in [\!:$ the lines of A_1, $\{1\} :\!]$ if and only if there exists X such that $x = \langle X, 1 \rangle$ and X is a line.

(19) For every x holds $x \in [\!:$ the directions of planes of A_1, $\{2\} :\!]$ if and only if there exists X such that $x = \langle$ the direction of $X, 2 \rangle$ and X is a plane.

Let us consider A_1. The projective points over A_1 yielding a non-empty set is defined as follows:

(Def.9) the projective points over A_1 = (the points of A_1) $\cup$ the directions of lines of A_1.

Let us consider A_1. The functor $L(A_1)$ yielding a non-empty set is defined as follows:

(Def.10) $L(A_1) = [\!:$ the lines of A_1, $\{1\} :\!] \cup [\!:$ the directions of planes of A_1, $\{2\} :\!]$.

Let us consider A_1. The functor $\mathbf{I}_{A_1}$ yielding a relation between the projective points over A_1 and $L(A_1)$ is defined by the condition (Def.11).

(Def.11) Given x, y. Then $\langle x, y \rangle \in \mathbf{I}_{A_1}$ if and only if there exists K such that K is a line and $y = \langle K, 1 \rangle$ but $x \in$ the points of A_1 and $x \in K$ or x = the direction of K or there exist K, X such that K is a line and X is a plane and x = the direction of K and $y = \langle$ the direction of $X, 2 \rangle$ and $K || X$.

Let us consider A_1. The incidence of directions of A_1 yields a relation between the directions of lines of A_1 and the directions of planes of A_1 and is defined as follows:

(Def.12) for all x, y holds $\langle x, y \rangle \in$ the incidence of directions of A_1 if and only if there exist A, X such that x = the direction of A and y = the direction of X and A is a line and X is a plane and $A || X$.

Let us consider A_1. The functor Inc-ProjSp(A_1) yielding a projective incidence structure is defined as follows:

(Def.13) Inc-ProjSp(A_1) = $\langle$the projective points over A_1, $L(A_1), \mathbf{I}_{A_1}\rangle$.

Let us consider A_1. The projective horizon of A_1 yielding a projective incidence structure is defined as follows:

(Def.14) the projective horizon of A_1 = $\langle$the directions of lines of A_1, the directions of planes of A_1, the incidence of directions of $A_1\rangle$.

We now state several propositions:

(20) For every x holds x is an element of the points of Inc-ProjSp(A_1) if and only if x is an element of the points of A_1 or there exists X such that $x = \text{the direction of } X$ and X is a line.

(21) x is an element of the points of the projective horizon of A_1 if and only if there exists X such that $x = \text{the direction of } X$ and X is a line.

(22) If x is an element of the points of the projective horizon of A_1, then x is an element of the points of Inc-ProjSp(A_1).

(23) For every x holds x is an element of the lines of Inc-ProjSp(A_1) if and only if there exists X such that $x = \langle X, 1\rangle$ and X is a line or $x = \langle \text{the direction of } X, 2\rangle$ and X is a plane.

(24) x is an element of the lines of the projective horizon of A_1 if and only if there exists X such that $x = \text{the direction of } X$ and X is a plane.

(25) If x is an element of the lines of the projective horizon of A_1, then $\langle x, 2\rangle$ is an element of the lines of Inc-ProjSp(A_1).

For simplicity we adopt the following rules: x will denote an element of the points of A_1, X, Y, X' will denote subsets of the points of A_1, a, p, q will denote elements of the points of Inc-ProjSp(A_1), and A will denote an element of the lines of Inc-ProjSp(A_1). We now state a number of propositions:

(26) If $x = a$ and $\langle X, 1\rangle = A$, then $a \mid A$ if and only if X is a line and $x \in X$.

(27) If $x = a$ and $\langle \text{the direction of } X, 2\rangle = A$ and X is a plane, then $a \nmid A$.

(28) If $a = \text{the direction of } Y$ and $\langle X, 1\rangle = A$ and Y is a line and X is a line, then $a \mid A$ if and only if $Y || X$.

(29) If $a = \text{the direction of } Y$ and $A = \langle \text{the direction of } X, 2\rangle$ and Y is a line and X is a plane, then $a \mid A$ if and only if $Y || X$.

(30) If X is a line and $a = \text{the direction of } X$ and $A = \langle X, 1\rangle$, then $a \mid A$.

(31) If X is a line and Y is a plane and $X \subseteq Y$ and $a = \text{the direction of } X$ and $A = \langle \text{the direction of } Y, 2\rangle$, then $a \mid A$.

(32) If Y is a plane and $X \subseteq Y$ and $X' \parallel X$ and $a = \text{the direction of } X'$ and $A = \langle \text{the direction of } Y, 2\rangle$, then $a \mid A$.

(33) If $A = \langle \text{the direction of } X, 2\rangle$ and X is a plane and $a \mid A$, then a is not an element of the points of A_1.

(34) If $A = \langle X, 1\rangle$ and X is a line and $p \mid A$ and p is not an element of the points of A_1, then $p = \text{the direction of } X$.

(35) If $A = \langle X, 1\rangle$ and X is a line and $p \mid A$ and $a \mid A$ and $a \neq p$ and p is not an element of the points of A_1, then a is an element of the points of A_1.

(36) For every element a of the points of the projective horizon of A_1 and for every element A of the lines of the projective horizon of A_1 such that $a = \text{the direction of } X$ and $A = \text{the direction of } Y$ and X is a line and Y is a plane holds $a \mid A$ if and only if $X || Y$.

(37) For every element a of the points of the projective horizon of A_1 and for every element a' of the points of Inc-ProjSp(A_1) and for every element A of the lines of the projective horizon of A_1 and for every element A' of the lines of Inc-ProjSp(A_1) such that $a' = a$ and $A' = \langle A, 2 \rangle$ holds $a \mid A$ if and only if $a' \mid A'$.

In the sequel P, Q denote elements of the lines of Inc-ProjSp(A_1). We now state several propositions:

(38) For all elements a, b of the points of the projective horizon of A_1 and for all elements A, K of the lines of the projective horizon of A_1 such that $a \mid A$ and $a \mid K$ and $b \mid A$ and $b \mid K$ holds $a = b$ or $A = K$.

(39) For every element A of the lines of the projective horizon of A_1 there exist elements a, b, c of the points of the projective horizon of A_1 such that $a \mid A$ and $b \mid A$ and $c \mid A$ and $a \neq b$ and $b \neq c$ and $c \neq a$.

(40) For every elements a, b of the points of the projective horizon of A_1 there exists an element A of the lines of the projective horizon of A_1 such that $a \mid A$ and $b \mid A$.

(41) For all elements x, y of the points of the projective horizon of A_1 and for every element X of the lines of Inc-ProjSp(A_1) such that $x \neq y$ and $\langle x, X \rangle \in$ the incidence of Inc-ProjSp(A_1) and $\langle y, X \rangle \in$ the incidence of Inc-ProjSp(A_1) there exists an element Y of the lines of the projective horizon of A_1 such that $X = \langle Y, 2 \rangle$.

(42) For every element x of the points of Inc-ProjSp(A_1) and for every element X of the lines of the projective horizon of A_1 such that $\langle x, \langle X, 2 \rangle \rangle \in$ the incidence of Inc-ProjSp(A_1) holds x is an element of the points of the projective horizon of A_1.

(43) If Y is a plane and X is a line and X' is a line and $X \subseteq Y$ and $X' \subseteq Y$ and $P = \langle X, 1 \rangle$ and $Q = \langle X', 1 \rangle$, then there exists q such that $q \mid P$ and $q \mid Q$.

(44) Let a, b, c, d, p be elements of the points of the projective horizon of A_1. Let M, N, P, Q be elements of the lines of the projective horizon of A_1. Suppose that

(i) $a \mid M$,
(ii) $b \mid M$,
(iii) $c \mid N$,
(iv) $d \mid N$,
(v) $p \mid M$,
(vi) $p \mid N$,
(vii) $a \mid P$,
(viii) $c \mid P$,
(ix) $b \mid Q$,
(x) $d \mid Q$,
(xi) $p \nmid P$,
(xii) $p \nmid Q$,

(xiii) $M \neq N$.

Then there exists an element q of the points of the projective horizon of A_1 such that $q \mid P$ and $q \mid Q$.

Let us consider A_1. Then Inc-ProjSp(A_1) is a projective space defined in terms of incidence.

Let A_1 be an affine plane. Then Inc-ProjSp(A_1) is a 2-dimensional projective space defined in terms of incidence.

The following propositions are true:

(45) If Inc-ProjSp(A_1) is 2-dimensional, then A_1 is an affine plane.

(46) If A_1 is not an affine plane, then the projective horizon of A_1 is a projective space defined in terms of incidence.

(47) If the projective horizon of A_1 is a projective space defined in terms of incidence, then A_1 is not an affine plane.

(48) Let M, N be subsets of the points of A_1. Let o, a, b, c, a', b', c' be elements of the points of A_1. Suppose that

(i) M is a line,
(ii) N is a line,
(iii) $M \neq N$,
(iv) $o \in M$,
(v) $o \in N$,
(vi) $o \neq a$,
(vii) $o \neq a'$,
(viii) $o \neq b$,
(ix) $o \neq b'$,
(x) $o \neq c$,
(xi) $o \neq c'$,
(xii) $a \in M$,
(xiii) $b \in M$,
(xiv) $c \in M$,
(xv) $a' \in N$,
(xvi) $b' \in N$,
(xvii) $c' \in N$,
(xviii) $a, b' \parallel b, a'$,
(xix) $b, c' \parallel c, b'$,
(xx) $a = b$ or $b = c$ or $a = c$.

Then $a, c' \parallel c, a'$.

(49) If Inc-ProjSp(A_1) is Pappian, then A_1 is Pappian.

(50) Let A, P, C be subsets of the points of A_1. Let o, a, b, c, a', b', c' be elements of the points of A_1. Suppose that

(i) $o \in A$,
(ii) $o \in P$,
(iii) $o \in C$,
(iv) $o \neq a$,

(v) $o \neq b$,
(vi) $o \neq c$,
(vii) $a \in A$,
(viii) $a' \in A$,
(ix) $b \in P$,
(x) $b' \in P$,
(xi) $c \in C$,
(xii) $c' \in C$,
(xiii) A is a line,
(xiv) P is a line,
(xv) C is a line,
(xvi) $A \neq P$,
(xvii) $A \neq C$,
(xviii) $a, b \parallel a', b'$,
(xix) $a, c \parallel a', c'$,
(xx) $o = a'$ or $a = a'$.

Then $b, c \parallel b', c'$.

(51) If Inc-ProjSp(A_1) is Desarguesian, then A_1 is Desarguesian.

(52) If Inc-ProjSp(A_1) is Fanoian, then A_1 is Fanoian.

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