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Planes in Affine Spaces [1]

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Summary. We introduce the notion of plane in affine space and investigate fundamental properties of them. Further we introduce the relation of parallelism defined for arbitrary subsets. In particular we are concerned with parallelisms which hold between lines and planes and between planes. We also define a function which assigns to every line and every point the unique line passing through the point and parallel to the given line. With the help of the introduced notions we prove that every at least 3-dimensional affine space is Desarguesian and translation.

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The articles [5], [1], [2], [3], and [4] provide the notation and terminology for this paper. We follow a convention: A_1 will be an affine space, a, b, c, d, a', b', c', p, q will be elements of the points of A_1, and A, C, K, M, N, P, Q, X, Y, Z will be subsets of the points of A_1. Let us consider A_1, X, Y. Then $X \cap Y$ is a subset of the points of A_1.

The following propositions are true:

(1) If $\mathbf{L}(p,a,a')$ or $\mathbf{L}(p,a',a)$ but $p \neq a$, then there exists b' such that $\mathbf{L}(p,b,b')$ and $a,b \parallel a',b'$.

(2) If $a,b \parallel A$ or $b,a \parallel A$ but $a \in A$, then $b \in A$.

(3) If $a,b \parallel A$ or $b,a \parallel A$ but $A \parallel K$ or $K \parallel A$, then $a,b \parallel K$ and $b,a \parallel K$.

(4) If $a,b \parallel A$ or $b,a \parallel A$ but $a,b \parallel c,d$ or $c,d \parallel a,b$ and $a \neq b$, then $c,d \parallel A$ and $d,c \parallel A$.

(5) If $a,b \parallel M$ or $b,a \parallel M$ but $a,b \parallel N$ or $b,a \parallel N$ and $a \neq b$, then $M \parallel N$ and $N \parallel M$.

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(6) If $a, b \parallel M$ or $b, a \parallel M$ but $c, d \parallel M$ or $d, c \parallel M$, then $a, b \parallel c, d$ and $a, b \parallel d, c$.

(7) If $A \parallel C$ or $C \parallel A$ but $a \neq b$ but $a, b \parallel c, d$ or $c, d \parallel a, b$ and $a \in A$ and $b \in A$ and $c \in C$, then $d \in C$.

(8) Suppose that
(i) $q \in M$,
(ii) $q \in N$,
(iii) $a \in M$,
(iv) $a' \in M$,
(v) $b \in N$,
(vi) $b' \in N$,
(vii) $q \neq a$,
(viii) $q \neq b$,
(ix) $M \neq N$,
(x) $a, b \parallel a', b'$ or $b, a \parallel b', a'$,
(xi) M is a line,
(xii) N is a line,
(xiii) $q = a'$.
Then $q = b'$.

(9) Suppose that
(i) $q \in M$,
(ii) $q \in N$,
(iii) $a \in M$,
(iv) $a' \in M$,
(v) $b \in N$,
(vi) $b' \in N$,
(vii) $q \neq a$,
(viii) $q \neq b$,
(ix) $M \neq N$,
(x) $a, b \parallel a', b'$ or $b, a \parallel b', a'$,
(xi) M is a line,
(xii) N is a line,
(xiii) $a = a'$.
Then $b = b'$.

(10) If $M \parallel N$ or $N \parallel M$ but $a \in M$ and $a' \in M$ and $b \in N$ and $b' \in N$ and $M \neq N$ but $a, b \parallel a', b'$ or $b, a \parallel b', a'$ and $a = a'$, then $b = b'$.

(11) There exists A such that $a \in A$ and $b \in A$ and A is a line.

(12) If A is a line, then there exists q such that $q \notin A$.

Let us consider A_1, K, P. The functor $\mathrm{Plane}(K, P)$ yielding a subset of the points of A_1 is defined by:

(Def.1) $\mathrm{Plane}(K, P) = \{a : \bigvee_b [a, b \parallel K \wedge b \in P]\}$.

Let us consider A_1, X. We say that X is a plane if and only if:

(Def.2) there exist K, P such that K is a line and P is a line and $K \nparallel P$ and $X = \mathrm{Plane}(K, P)$.

We now state a number of propositions:

(13) If K is not a line, then $\mathrm{Plane}(K, P) = \emptyset$.

(14) If K is a line, then $P \subseteq \mathrm{Plane}(K, P)$.

(15) If $K \parallel P$, then $\mathrm{Plane}(K, P) = P$.

(16) If $K \parallel M$, then $\mathrm{Plane}(K, P) = \mathrm{Plane}(M, P)$.

(17) Suppose that

(i) $p \in M$,
(ii) $a \in M$,
(iii) $b \in M$,
(iv) $p \in N$,
(v) $a' \in N$,
(vi) $b' \in N$,
(vii) $p \notin P$,
(viii) $p \notin Q$,
(ix) $M \neq N$,
(x) $a \in P$,
(xi) $a' \in P$,
(xii) $b \in Q$,
(xiii) $b' \in Q$,
(xiv) M is a line,
(xv) N is a line,
(xvi) P is a line,
(xvii) Q is a line.

Then $P \parallel Q$ or there exists q such that $q \in P$ and $q \in Q$.

(18) Suppose $a \in M$ and $b \in M$ and $a' \in N$ and $b' \in N$ and $a \in P$ and $a' \in P$ and $b \in Q$ and $b' \in Q$ and $M \neq N$ and $M \parallel N$ and P is a line and Q is a line. Then $P \parallel Q$ or there exists q such that $q \in P$ and $q \in Q$.

(19) If X is a plane and $a \in X$ and $b \in X$ and $a \neq b$, then $\mathrm{Line}(a, b) \subseteq X$.

(20) If K is a line and P is a line and Q is a line and $K \nparallel P$ and $K \nparallel Q$ and $Q \subseteq \mathrm{Plane}(K, P)$, then $\mathrm{Plane}(K, Q) = \mathrm{Plane}(K, P)$.

(21) If K is a line and P is a line and Q is a line and $K \nparallel P$ and $Q \subseteq \mathrm{Plane}(K, P)$, then $P \parallel Q$ or there exists q such that $q \in P$ and $q \in Q$.

(22) If X is a plane and M is a line and N is a line and $M \subseteq X$ and $N \subseteq X$, then $M \parallel N$ or there exists q such that $q \in M$ and $q \in N$.

(23) If X is a plane and $a \in X$ and $M \subseteq X$ and $a \in N$ but $M \parallel N$ or $N \parallel M$, then $N \subseteq X$.

(24) If X is a plane and Y is a plane and $a \in X$ and $b \in X$ and $a \in Y$ and $b \in Y$ and $X \neq Y$ and $a \neq b$, then $X \cap Y$ is a line.

(25) If X is a plane and Y is a plane and $a \in X$ and $b \in X$ and $c \in X$ and $a \in Y$ and $b \in Y$ and $c \in Y$ and not $\mathbf{L}(a, b, c)$, then $X = Y$.

(26) If X is a plane and Y is a plane and M is a line and N is a line and $M \subseteq X$ and $N \subseteq X$ and $M \subseteq Y$ and $N \subseteq Y$ and $M \neq N$, then $X = Y$.

Let us consider A_1, a, K. Let us assume that K is a line. The functor $a \cdot K$ yields a subset of the points of A_1 and is defined by:

(Def.3) $a \in a \cdot K$ and $K \parallel a \cdot K$.

We now state several propositions:

(27) If A is a line, then $a \cdot A$ is a line.

(28) If X is a plane and M is a line and $a \in X$ and $M \subseteq X$, then $a \cdot M \subseteq X$.

(29) If X is a plane and $a \in X$ and $b \in X$ and $c \in X$ and $a, b \parallel c, d$ and $a \neq b$, then $d \in X$.

(30) If A is a line, then $a \in A$ if and only if $a \cdot A = A$.

(31) If A is a line, then $a \cdot A = a \cdot (q \cdot A)$.

(32) If $K \parallel M$, then $a \cdot K = a \cdot M$.

Let us consider A_1, X, Y. The predicate $X || Y$ is defined by:

(Def.4) for all a, A such that $a \in Y$ and A is a line and $A \subseteq X$ holds $a \cdot A \subseteq Y$.

Next we state a number of propositions:

(33) If $X \subseteq Y$ but X is a line and Y is a line or X is a plane and Y is a plane, then $X = Y$.

(34) If X is a plane, then there exist a, b, c such that $a \in X$ and $b \in X$ and $c \in X$ and not $\mathbf{L}(a, b, c)$.

(35) If M is a line and X is a plane and $M \subseteq X$, then there exists q such that $q \in X$ and $q \notin M$.

(36) For all a, A such that A is a line there exists X such that $a \in X$ and $A \subseteq X$ and X is a plane.

(37) There exists X such that $a \in X$ and $b \in X$ and $c \in X$ and X is a plane.

(38) If $q \in M$ and $q \in N$ and M is a line and N is a line, then there exists X such that $M \subseteq X$ and $N \subseteq X$ and X is a plane.

(39) If $M \parallel N$, then there exists X such that $M \subseteq X$ and $N \subseteq X$ and X is a plane.

(40) If M is a line and N is a line, then $M \parallel N$ if and only if $M || N$.

(41) If M is a line and X is a plane, then $M || X$ if and only if there exists N such that $N \subseteq X$ but $M \parallel N$ or $N \parallel M$.

(42) If M is a line and X is a plane and $M \subseteq X$, then $M || X$.

(43) If A is a line and X is a plane and $a \in A$ and $a \in X$ and $A || X$, then $A \subseteq X$.

Let us consider A_1, K, M, N. We say that K, M, N are coplanar if and only if:

(Def.5) there exists X such that $K \subseteq X$ and $M \subseteq X$ and $N \subseteq X$ and X is a plane.

The following propositions are true:

(44) If K, M, N are coplanar, then K, N, M are coplanar and M, K, N are coplanar and M, N, K are coplanar and N, K, M are coplanar and N, M, K are coplanar.

(45) If A is a line and K is a line and M is a line and N is a line and M, N, K are coplanar and M, N, A are coplanar and $M \neq N$, then M, K, A are coplanar.

(46) If K is a line and M is a line and X is a plane and $K \subseteq X$ and $M \subseteq X$ and $K \neq M$, then K, M, A are coplanar if and only if $A \subseteq X$.

(47) If $q \in K$ and $q \in M$ and K is a line and M is a line, then K, M, M are coplanar and M, K, M are coplanar and M, M, K are coplanar.

(48) If A_1 is not an affine plane and X is a plane, then there exists q such that $q \notin X$.

(49) Suppose that
(i) A_1 is not an affine plane,
(ii) $q \in A$,
(iii) $q \in P$,
(iv) $q \in C$,
(v) $q \neq a$,
(vi) $q \neq b$,
(vii) $q \neq c$,
(viii) $a \in A$,
(ix) $a' \in A$,
(x) $b \in P$,
(xi) $b' \in P$,
(xii) $c \in C$,
(xiii) $c' \in C$,
(xiv) A is a line,
(xv) P is a line,
(xvi) C is a line,
(xvii) $A \neq P$,
(xviii) $A \neq C$,
(xix) $a, b \parallel a', b'$,
(xx) $a, c \parallel a', c'$.

Then $b, c \parallel b', c'$.

(50) If A_1 is not an affine plane, then A_1 is Desarguesian.

(51) Suppose that
(i) A_1 is not an affine plane,
(ii) $A \parallel P$,
(iii) $A \parallel C$,
(iv) $a \in A$,
(v) $a' \in A$,
(vi) $b \in P$,
(vii) $b' \in P$,
(viii) $c \in C$,

(ix) $c' \in C$,
(x) A is a line,
(xi) P is a line,
(xii) C is a line,
(xiii) $A \neq P$,
(xiv) $A \neq C$,
(xv) $a, b \parallel a', b'$,
(xvi) $a, c \parallel a', c'$.

Then $b, c \parallel b', c'$.

(52) If A_1 is not an affine plane, then A_1 is translation.

(53) If A_1 is an affine plane and not $\mathbf{L}(a, b, c)$, then there exists c' such that $a, c \parallel a', c'$ and $b, c \parallel b', c'$.

(54) If not $\mathbf{L}(a, b, c)$ and $a' \neq b'$ and $a, b \parallel a', b'$, then there exists c' such that $a, c \parallel a', c'$ and $b, c \parallel b', c'$.

(55) Suppose X is a plane and Y is a plane. Then $X||Y$ if and only if there exist A, P, M, N such that $A \nparallel P$ and $A \subseteq X$ and $P \subseteq X$ and $M \subseteq Y$ and $N \subseteq Y$ but $A \parallel M$ or $M \parallel A$ but $P \parallel N$ or $N \parallel P$.

(56) If $A \parallel M$ and $M||X$, then $A||X$.

(57) If X is a plane, then $X||X$.

(58) If X is a plane and Y is a plane and $X||Y$, then $Y||X$.

(59) If X is a plane, then $X \neq \emptyset$.

(60) If $X||Y$ and $Y||Z$ and $Y \neq \emptyset$, then $X||Z$.

(61) If X is a plane and Y is a plane and Z is a plane but $X||Y$ and $Y||Z$ or $X||Y$ and $Z||Y$ or $Y||X$ and $Y||Z$ or $Y||X$ and $Z||Y$, then $X||Z$ and $Z||X$.

(62) If X is a plane and Y is a plane and $a \in X$ and $a \in Y$ and $X||Y$, then $X = Y$.

(63) If X is a plane and Y is a plane and Z is a plane and $X||Y$ and $X \neq Y$ and $a \in X \cap Z$ and $b \in X \cap Z$ and $c \in Y \cap Z$ and $d \in Y \cap Z$, then $a, b \parallel c, d$.

(64) Suppose X is a plane and Y is a plane and Z is a plane and $X||Y$ and $a \in X \cap Z$ and $b \in X \cap Z$ and $c \in Y \cap Z$ and $d \in Y \cap Z$ and $X \neq Y$ and $a \neq b$ and $c \neq d$. Then $X \cap Z \parallel Y \cap Z$.

(65) For all a, X such that X is a plane there exists Y such that $a \in Y$ and $X||Y$ and Y is a plane.

Let us consider A_1, a, X. Let us assume that X is a plane. The functor $a + X$ yields a subset of the points of A_1 and is defined as follows:

(Def.6) $a \in a + X$ and $X||a + X$ and $a + X$ is a plane.

Next we state four propositions:

(66) If X is a plane, then $a \in X$ if and only if $a + X = X$.

(67) If X is a plane, then $a + X = a + (q + X)$.

(68) If A is a line and X is a plane and $A||X$, then $a \cdot A \subseteq a + X$.

(69) If X is a plane and Y is a plane and $X || Y$, then $a + X = a + Y$.

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