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# Ordered Rings - Part III

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**Summary.** This series of papers is devoted to the notion of the ordered ring, and one of its most important cases: the notion of ordered field. It follows the results of [6]. The idea of the notion of order in the ring is based on that of positive cone i.e. the set of positive elements. Positive cone has to contain at least squares of all elements, and has to be closed under sum and product. Therefore the key notions of this theory are that of square, sum of squares, product of squares, etc. and finally elements generated from squares by means of sums and products. Part III contains the classification of products of such elements.

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The papers [1], [2], [7], [3], [4], and [5] provide the terminology and notation for this paper. In the sequel $R$ will denote a field structure and $x$, $y$ will denote scalars of $R$. Next we state a number of propositions:

(1) If $x$ is a square and $y$ is a square, then $x \cdot y$ is a product of squares.

(2) If $x$ is a product of squares and $y$ is a square, then $x \cdot y$ is a product of squares.

(3) If $x$ is a square and $y$ is a product of squares or $x$ is a square and $y$ is a amalgam of squares, then $x \cdot y$ is a amalgam of squares.

(4) If $x$ is a product of squares and $y$ is a product of squares or $x$ is a product of squares and $y$ is a amalgam of squares, then $x \cdot y$ is a amalgam of squares.

(5) If $x$ is a amalgam of squares and $y$ is a square or $x$ is a amalgam of squares and $y$ is a product of squares or $x$ is a amalgam of squares and $y$ is a amalgam of squares, then $x \cdot y$ is a amalgam of squares.

(6) If $x$ is a square and $y$ is a sum of squares or $x$ is a square and $y$ is a sum of products of squares or $x$ is a square and $y$ is a sum of amalgams of squares or $x$ is a square and $y$ is generated from squares, then $x \cdot y$ is generated from squares.




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(7) If $x$ is a sum of squares and $y$ is a square or $x$ is a sum of squares and $y$ is a sum of squares or $x$ is a sum of squares and $y$ is a product of squares or $x$ is a sum of squares and $y$ is a sum of products of squares or $x$ is a sum of squares and $y$ is a amalgam of squares or $x$ is a sum of squares and $y$ is a sum of amalgams of squares or $x$ is a sum of squares and $y$ is generated from squares, then $x \cdot y$ is generated from squares.

(8) If $x$ is a product of squares and $y$ is a sum of squares or $x$ is a product of squares and $y$ is a sum of products of squares or $x$ is a product of squares and $y$ is a sum of amalgams of squares or $x$ is a product of squares and $y$ is generated from squares, then $x \cdot y$ is generated from squares.

(9) If $x$ is a sum of products of squares and $y$ is a square or $x$ is a sum of products of squares and $y$ is a sum of squares or $x$ is a sum of products of squares and $y$ is a product of squares or $x$ is a sum of products of squares and $y$ is a sum of products of squares or $x$ is a sum of products of squares and $y$ is a amalgam of squares or $x$ is a sum of products of squares and $y$ is a sum of amalgams of squares or $x$ is a sum of products of squares and $y$ is generated from squares, then $x \cdot y$ is generated from squares.

(10) If $x$ is a amalgam of squares and $y$ is a sum of squares or $x$ is a amalgam of squares and $y$ is a sum of products of squares or $x$ is a amalgam of squares and $y$ is a sum of amalgams of squares or $x$ is a amalgam of squares and $y$ is generated from squares, then $x \cdot y$ is generated from squares.

(11) If $x$ is a sum of amalgams of squares and $y$ is a square or $x$ is a sum of amalgams of squares and $y$ is a sum of squares or $x$ is a sum of amalgams of squares and $y$ is a product of squares or $x$ is a sum of amalgams of squares and $y$ is a sum of products of squares or $x$ is a sum of amalgams of squares and $y$ is a amalgam of squares or $x$ is a sum of amalgams of squares and $y$ is a sum of amalgams of squares or $x$ is a sum of amalgams of squares and $y$ is generated from squares, then $x \cdot y$ is generated from squares.

(12) If $x$ is generated from squares and $y$ is a square or $x$ is generated from squares and $y$ is a sum of squares or $x$ is generated from squares and $y$ is a product of squares or $x$ is generated from squares and $y$ is a sum of products of squares or $x$ is generated from squares and $y$ is a amalgam of squares or $x$ is generated from squares and $y$ is a sum of amalgams of squares or $x$ is generated from squares and $y$ is generated from squares, then $x \cdot y$ is generated from squares.

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