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One-Side Limits of a Real Function at a Point

Jarosław Kotowicz[1]
Warsaw University
Białystok

Summary. We introduce the left-side and the right-side limit of a real function at a point. We prove a few properties of the operations on the proper and improper one-side limits and show that Cauchy and Heine characterizations of one-side limit are equivalent.

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The articles [15], [4], [1], [2], [13], [11], [5], [7], [12], [14], [3], [8], [9], [10], and [6] provide the terminology and notation for this paper. For simplicity we adopt the following convention: r, r_1, r_2, g, g_1, x_0 will be real numbers, n, k will be natural numbers, s_1 will be a sequence of real numbers, and f, f_1, f_2 will be partial functions from $\mathbb{R}$ to $\mathbb{R}$. We now state several propositions:

(1) If s_1 is convergent and $r < \lim s_1$, then there exists n such that for every k such that $n \leq k$ holds $r < s_1(k)$.

(2) If s_1 is convergent and $\lim s_1 < r$, then there exists n such that for every k such that $n \leq k$ holds $s_1(k) < r$.

(3) If $0 < r_2$ and $]r_1 - r_2, r_1[\subseteq \operatorname{dom} f$, then for every r such that $r < r_1$ there exists g such that $r < g$ and $g < r_1$ and $g \in \operatorname{dom} f$.

(4) If $0 < r_2$ and $]r_1, r_1 + r_2[\subseteq \operatorname{dom} f$, then for every r such that $r_1 < r$ there exists g such that $g < r$ and $r_1 < g$ and $g \in \operatorname{dom} f$.

(5) If for every n holds $x_0 - \frac{1}{n+1} < s_1(n)$ and $s_1(n) < x_0$ and $s_1(n) \in \operatorname{dom} f$, then s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]-\infty, x_0[$.

(6) If for every n holds $x_0 < s_1(n)$ and $s_1(n) < x_0 + \frac{1}{n+1}$ and $s_1(n) \in \operatorname{dom} f$, then s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]x_0, +\infty[$.

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We now define several new predicates. Let us consider f, x_0. We say that f is left convergent in x_0 if and only if:

(Def.1) (i) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$,

(ii) there exists g such that for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]-\infty, x_0[$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

We say that f is left divergent to $+\infty$ in x_0 if and only if:

(Def.2) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]-\infty, x_0[$ holds $f \cdot s_1$ is divergent to $+\infty$.

We say that f is left divergent to $-\infty$ in x_0 if and only if:

(Def.3) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]-\infty, x_0[$ holds $f \cdot s_1$ is divergent to $-\infty$.

We say that f is right convergent in x_0 if and only if:

(Def.4) (i) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$,

(ii) there exists g such that for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]x_0, +\infty[$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

We say that f is right divergent to $+\infty$ in x_0 if and only if:

(Def.5) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]x_0, +\infty[$ holds $f \cdot s_1$ is divergent to $+\infty$.

We say that f is right divergent to $-\infty$ in x_0 if and only if:

(Def.6) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]x_0, +\infty[$ holds $f \cdot s_1$ is divergent to $-\infty$.

We now state a number of propositions:

(7) f is left convergent in x_0 if and only if the following conditions are satisfied:

(i) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$,

(ii) there exists g such that for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]-\infty, x_0[$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

(8) f is left divergent to $+\infty$ in x_0 if and only if for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]-\infty, x_0[$ holds $f \cdot s_1$ is divergent to $+\infty$.

(9) f is left divergent to $-\infty$ in x_0 if and only if for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]-\infty, x_0[$ holds $f \cdot s_1$ is divergent to $-\infty$.

(10) f is right convergent in x_0 if and only if the following conditions are satisfied:

(i) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$,

(ii) there exists g such that for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]x_0, +\infty[$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

(11) f is right divergent to $+\infty$ in x_0 if and only if for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]x_0, +\infty[$ holds $f \cdot s_1$ is divergent to $+\infty$.

(12) f is right divergent to $-\infty$ in x_0 if and only if for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$ and for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\operatorname{rng} s_1 \subseteq \operatorname{dom} f \cap]x_0, +\infty[$ holds $f \cdot s_1$ is divergent to $-\infty$.

(13) f is left convergent in x_0 if and only if the following conditions are satisfied:

(i) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$,

(ii) there exists g such that for every g_1 such that $0 < g_1$ there exists r such that $r < x_0$ and for every r_1 such that $r < r_1$ and $r_1 < x_0$ and $r_1 \in \operatorname{dom} f$ holds $|f(r_1) - g| < g_1$.

(14) f is left divergent to $+\infty$ in x_0 if and only if the following conditions are satisfied:

(i) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$,

(ii) for every g_1 there exists r such that $r < x_0$ and for every r_1 such that $r < r_1$ and $r_1 < x_0$ and $r_1 \in \operatorname{dom} f$ holds $g_1 < f(r_1)$.

(15) f is left divergent to $-\infty$ in x_0 if and only if the following conditions are satisfied:

(i) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$,

(ii) for every g_1 there exists r such that $r < x_0$ and for every r_1 such that $r < r_1$ and $r_1 < x_0$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < g_1$.

(16) f is right convergent in x_0 if and only if the following conditions are satisfied:

(i) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$,

(ii) there exists g such that for every g_1 such that $0 < g_1$ there exists r such that $x_0 < r$ and for every r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \operatorname{dom} f$

holds $|f(r_1) - g| < g_1$.

(17) f is right divergent to $+\infty$ in x_0 if and only if the following conditions are satisfied:

(i) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$,

(ii) for every g_1 there exists r such that $x_0 < r$ and for every r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \operatorname{dom} f$ holds $g_1 < f(r_1)$.

(18) f is right divergent to $-\infty$ in x_0 if and only if the following conditions are satisfied:

(i) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$,

(ii) for every g_1 there exists r such that $x_0 < r$ and for every r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \operatorname{dom} f$ holds $f(r_1) < g_1$.

(19) If f_1 is left divergent to $+\infty$ in x_0 and f_2 is left divergent to $+\infty$ in x_0 and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then $f_1 + f_2$ is left divergent to $+\infty$ in x_0 and $f_1 f_2$ is left divergent to $+\infty$ in x_0.

(20) If f_1 is left divergent to $-\infty$ in x_0 and f_2 is left divergent to $-\infty$ in x_0 and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then $f_1 + f_2$ is left divergent to $-\infty$ in x_0 and $f_1 f_2$ is left divergent to $+\infty$ in x_0.

(21) If f_1 is right divergent to $+\infty$ in x_0 and f_2 is right divergent to $+\infty$ in x_0 and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then $f_1 + f_2$ is right divergent to $+\infty$ in x_0 and $f_1 f_2$ is right divergent to $+\infty$ in x_0.

(22) If f_1 is right divergent to $-\infty$ in x_0 and f_2 is right divergent to $-\infty$ in x_0 and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$, then $f_1 + f_2$ is right divergent to $-\infty$ in x_0 and $f_1 f_2$ is right divergent to $+\infty$ in x_0.

(23) If f_1 is left divergent to $+\infty$ in x_0 and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom}(f_1 + f_2)$ and there exists r such that $0 < r$ and f_2 is lower bounded on $]x_0 - r, x_0[$, then $f_1 + f_2$ is left divergent to $+\infty$ in x_0.

(24) Suppose that

(i) f_1 is left divergent to $+\infty$ in x_0,

(ii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom}(f_1 f_2)$,

(iii) there exist r, r_1 such that $0 < r$ and $0 < r_1$ and for every g such that $g \in \operatorname{dom} f_2 \cap\,]x_0 - r, x_0[$ holds $r_1 \leq f_2(g)$.

Then $f_1 f_2$ is left divergent to $+\infty$ in x_0.

(25) If f_1 is right divergent to $+\infty$ in x_0 and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom}(f_1 + f_2)$ and there exists r such that $0 < r$ and f_2 is lower bounded on $]x_0, x_0 + r[$, then

$f_1 + f_2$ is right divergent to $+\infty$ in x_0.

(26) Suppose that

(i) f_1 is right divergent to $+\infty$ in x_0,

(ii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom}(f_1 f_2)$,

(iii) there exist r, r_1 such that $0 < r$ and $0 < r_1$ and for every g such that $g \in \operatorname{dom} f_2 \cap]x_0, x_0 + r[$ holds $r_1 \leq f_2(g)$.

Then $f_1 f_2$ is right divergent to $+\infty$ in x_0.

(27) (i) If f is left divergent to $+\infty$ in x_0 and $r > 0$, then rf is left divergent to $+\infty$ in x_0,

(ii) if f is left divergent to $+\infty$ in x_0 and $r < 0$, then rf is left divergent to $-\infty$ in x_0,

(iii) if f is left divergent to $-\infty$ in x_0 and $r > 0$, then rf is left divergent to $-\infty$ in x_0,

(iv) if f is left divergent to $-\infty$ in x_0 and $r < 0$, then rf is left divergent to $+\infty$ in x_0.

(28) (i) If f is right divergent to $+\infty$ in x_0 and $r > 0$, then rf is right divergent to $+\infty$ in x_0,

(ii) if f is right divergent to $+\infty$ in x_0 and $r < 0$, then rf is right divergent to $-\infty$ in x_0,

(iii) if f is right divergent to $-\infty$ in x_0 and $r > 0$, then rf is right divergent to $-\infty$ in x_0,

(iv) if f is right divergent to $-\infty$ in x_0 and $r < 0$, then rf is right divergent to $+\infty$ in x_0.

(29) If f is left divergent to $+\infty$ in x_0 or f is left divergent to $-\infty$ in x_0, then $|f|$ is left divergent to $+\infty$ in x_0.

(30) If f is right divergent to $+\infty$ in x_0 or f is right divergent to $-\infty$ in x_0, then $|f|$ is right divergent to $+\infty$ in x_0.

(31) If there exists r such that $0 < r$ and f is non-decreasing on $]x_0 - r, x_0[$ and f is not upper bounded on $]x_0 - r, x_0[$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$, then f is left divergent to $+\infty$ in x_0.

(32) If there exists r such that $0 < r$ and f is increasing on $]x_0 - r, x_0[$ and f is not upper bounded on $]x_0 - r, x_0[$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$, then f is left divergent to $+\infty$ in x_0.

(33) If there exists r such that $0 < r$ and f is non-increasing on $]x_0 - r, x_0[$ and f is not lower bounded on $]x_0 - r, x_0[$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$, then f is left divergent to $-\infty$ in x_0.

(34) If there exists r such that $0 < r$ and f is decreasing on $]x_0 - r, x_0[$ and f is not lower bounded on $]x_0 - r, x_0[$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$, then f is left

divergent to $-\infty$ in x_0.

(35) If there exists r such that $0 < r$ and f is non-increasing on $]x_0, x_0 + r[$ and f is not upper bounded on $]x_0, x_0+r[$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$, then f is right divergent to $+\infty$ in x_0.

(36) If there exists r such that $0 < r$ and f is decreasing on $]x_0, x_0 + r[$ and f is not upper bounded on $]x_0, x_0 + r[$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$, then f is right divergent to $+\infty$ in x_0.

(37) If there exists r such that $0 < r$ and f is non-decreasing on $]x_0, x_0 + r[$ and f is not lower bounded on $]x_0, x_0+r[$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$, then f is right divergent to $-\infty$ in x_0.

Next we state several propositions:

(38) If there exists r such that $0 < r$ and f is increasing on $]x_0, x_0 + r[$ and f is not lower bounded on $]x_0, x_0 + r[$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$, then f is right divergent to $-\infty$ in x_0.

(39) Suppose that
(i) f_1 is left divergent to $+\infty$ in x_0,
(ii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$,
(iii) there exists r such that $0 < r$ and $\operatorname{dom} f \cap]x_0 - r, x_0[\subseteq \operatorname{dom} f_1 \cap]x_0 - r, x_0[$ and for every g such that $g \in \operatorname{dom} f \cap]x_0 - r, x_0[$ holds $f_1(g) \leq f(g)$.

Then f is left divergent to $+\infty$ in x_0.

(40) Suppose that
(i) f_1 is left divergent to $-\infty$ in x_0,
(ii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$,
(iii) there exists r such that $0 < r$ and $\operatorname{dom} f \cap]x_0 - r, x_0[\subseteq \operatorname{dom} f_1 \cap]x_0 - r, x_0[$ and for every g such that $g \in \operatorname{dom} f \cap]x_0 - r, x_0[$ holds $f(g) \leq f_1(g)$.

Then f is left divergent to $-\infty$ in x_0.

(41) Suppose that
(i) f_1 is right divergent to $+\infty$ in x_0,
(ii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$,
(iii) there exists r such that $0 < r$ and $\operatorname{dom} f \cap]x_0, x_0+r[\subseteq \operatorname{dom} f_1 \cap]x_0, x_0+r[$ and for every g such that $g \in \operatorname{dom} f \cap]x_0, x_0 + r[$ holds $f_1(g) \leq f(g)$.

Then f is right divergent to $+\infty$ in x_0.

(42) Suppose that
(i) f_1 is right divergent to $-\infty$ in x_0,
(ii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$,

(iii) there exists r such that $0 < r$ and $\mathrm{dom}\, f \cap]x_0, x_0+r[\subseteq \mathrm{dom}\, f_1 \cap]x_0, x_0+r[$ and for every g such that $g \in \mathrm{dom}\, f \cap]x_0, x_0+r[$ holds $f(g) \leq f_1(g)$.
Then f is right divergent to $-\infty$ in x_0.

(43) If f_1 is left divergent to $+\infty$ in x_0 and there exists r such that $0 < r$ and $]x_0-r, x_0[\subseteq \mathrm{dom}\, f \cap \mathrm{dom}\, f_1$ and for every g such that $g \in]x_0-r, x_0[$ holds $f_1(g) \leq f(g)$, then f is left divergent to $+\infty$ in x_0.

(44) If f_1 is left divergent to $-\infty$ in x_0 and there exists r such that $0 < r$ and $]x_0-r, x_0[\subseteq \mathrm{dom}\, f \cap \mathrm{dom}\, f_1$ and for every g such that $g \in]x_0-r, x_0[$ holds $f(g) \leq f_1(g)$, then f is left divergent to $-\infty$ in x_0.

(45) If f_1 is right divergent to $+\infty$ in x_0 and there exists r such that $0 < r$ and $]x_0, x_0+r[\subseteq \mathrm{dom}\, f \cap \mathrm{dom}\, f_1$ and for every g such that $g \in]x_0, x_0+r[$ holds $f_1(g) \leq f(g)$, then f is right divergent to $+\infty$ in x_0.

(46) If f_1 is right divergent to $-\infty$ in x_0 and there exists r such that $0 < r$ and $]x_0, x_0+r[\subseteq \mathrm{dom}\, f \cap \mathrm{dom}\, f_1$ and for every g such that $g \in]x_0, x_0+r[$ holds $f(g) \leq f_1(g)$, then f is right divergent to $-\infty$ in x_0.

Let us consider f, x_0. Let us assume that f is left convergent in x_0. The functor $\lim_{x_0^-} f$ yields a real number and is defined by:

(Def.7) for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f \cap]-\infty, x_0[$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = \lim_{x_0^-} f$.

Let us consider f, x_0. Let us assume that f is right convergent in x_0. The functor $\lim_{x_0^+} f$ yields a real number and is defined by:

(Def.8) for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f \cap]x_0, +\infty[$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = \lim_{x_0^+} f$.

One can prove the following propositions:

(47) If f is left convergent in x_0, then $\lim_{x_0^-} f = g$ if and only if for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f \cap]-\infty, x_0[$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

(48) If f is right convergent in x_0, then $\lim_{x_0^+} f = g$ if and only if for every s_1 such that s_1 is convergent and $\lim s_1 = x_0$ and $\mathrm{rng}\, s_1 \subseteq \mathrm{dom}\, f \cap]x_0, +\infty[$ holds $f \cdot s_1$ is convergent and $\lim(f \cdot s_1) = g$.

(49) If f is left convergent in x_0, then $\lim_{x_0^-} f = g$ if and only if for every g_1 such that $0 < g_1$ there exists r such that $r < x_0$ and for every r_1 such that $r < r_1$ and $r_1 < x_0$ and $r_1 \in \mathrm{dom}\, f$ holds $|f(r_1) - g| < g_1$.

(50) If f is right convergent in x_0, then $\lim_{x_0^+} f = g$ if and only if for every g_1 such that $0 < g_1$ there exists r such that $x_0 < r$ and for every r_1 such that $r_1 < r$ and $x_0 < r_1$ and $r_1 \in \mathrm{dom}\, f$ holds $|f(r_1) - g| < g_1$.

(51) If f is left convergent in x_0, then rf is left convergent in x_0 and $\lim_{x_0^-}(rf) = r \cdot (\lim_{x_0^-} f)$.

(52) If f is left convergent in x_0, then $-f$ is left convergent in x_0 and $\lim_{x_0^-}(-f) = -\lim_{x_0^-} f$.

(53) Suppose f_1 is left convergent in x_0 and f_2 is left convergent in x_0 and for every r such that $r < x_0$ there exists g such that $r < g$ and $g <$

x_0 and $g \in \operatorname{dom}(f_1 + f_2)$. Then $f_1 + f_2$ is left convergent in x_0 and $\lim_{x_0^-}(f_1 + f_2) = \lim_{x_0^-} f_1 + \lim_{x_0^-} f_2$.

(54) Suppose f_1 is left convergent in x_0 and f_2 is left convergent in x_0 and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom}(f_1 - f_2)$. Then $f_1 - f_2$ is left convergent in x_0 and $\lim_{x_0^-}(f_1 - f_2) = \lim_{x_0^-} f_1 - \lim_{x_0^-} f_2$.

(55) If f is left convergent in x_0 and $f^{-1}\{0\} = \emptyset$ and $\lim_{x_0^-} f \neq 0$, then $\frac{1}{f}$ is left convergent in x_0 and $\lim_{x_0^-} \frac{1}{f} = (\lim_{x_0^-} f)^{-1}$.

(56) If f is left convergent in x_0, then $|f|$ is left convergent in x_0 and $\lim_{x_0^-} |f| = |\lim_{x_0^-} f|$.

(57) Suppose f is left convergent in x_0 and $\lim_{x_0^-} f \neq 0$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$. Then $\frac{1}{f}$ is left convergent in x_0 and $\lim_{x_0^-} \frac{1}{f} = (\lim_{x_0^-} f)^{-1}$.

(58) Suppose f_1 is left convergent in x_0 and f_2 is left convergent in x_0 and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom}(f_1 f_2)$. Then $f_1 f_2$ is left convergent in x_0 and $\lim_{x_0^-}(f_1 f_2) = (\lim_{x_0^-} f_1) \cdot (\lim_{x_0^-} f_2)$.

(59) Suppose f_1 is left convergent in x_0 and f_2 is left convergent in x_0 and $\lim_{x_0^-} f_2 \neq 0$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} \frac{f_1}{f_2}$. Then $\frac{f_1}{f_2}$ is left convergent in x_0 and $\lim_{x_0^-} \frac{f_1}{f_2} = \frac{\lim_{x_0^-} f_1}{\lim_{x_0^-} f_2}$.

(60) If f is right convergent in x_0, then rf is right convergent in x_0 and $\lim_{x_0^+}(rf) = r \cdot (\lim_{x_0^+} f)$.

(61) If f is right convergent in x_0, then $-f$ is right convergent in x_0 and $\lim_{x_0^+}(-f) = -\lim_{x_0^+} f$.

(62) Suppose f_1 is right convergent in x_0 and f_2 is right convergent in x_0 and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom}(f_1 + f_2)$. Then $f_1 + f_2$ is right convergent in x_0 and $\lim_{x_0^+}(f_1 + f_2) = \lim_{x_0^+} f_1 + \lim_{x_0^+} f_2$.

(63) Suppose f_1 is right convergent in x_0 and f_2 is right convergent in x_0 and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom}(f_1 - f_2)$. Then $f_1 - f_2$ is right convergent in x_0 and $\lim_{x_0^+}(f_1 - f_2) = \lim_{x_0^+} f_1 - \lim_{x_0^+} f_2$.

(64) If f is right convergent in x_0 and $f^{-1}\{0\} = \emptyset$ and $\lim_{x_0^+} f \neq 0$, then $\frac{1}{f}$ is right convergent in x_0 and $\lim_{x_0^+} \frac{1}{f} = (\lim_{x_0^+} f)^{-1}$.

(65) If f is right convergent in x_0, then $|f|$ is right convergent in x_0 and $\lim_{x_0^+} |f| = |\lim_{x_0^+} f|$.

(66) Suppose f is right convergent in x_0 and $\lim_{x_0^+} f \neq 0$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$. Then $\frac{1}{f}$ is right convergent in x_0 and $\lim_{x_0^+} \frac{1}{f} = (\lim_{x_0^+} f)^{-1}$.

(67) Suppose f_1 is right convergent in x_0 and f_2 is right convergent in x_0

and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_1 f_2)$. Then $f_1 f_2$ is right convergent in x_0 and $\lim_{x_0^+}(f_1 f_2) = (\lim_{x_0^+} f_1) \cdot (\lim_{x_0^+} f_2)$.

(68) Suppose f_1 is right convergent in x_0 and f_2 is right convergent in x_0 and $\lim_{x_0^+} f_2 \neq 0$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}\, \frac{f_1}{f_2}$. Then $\frac{f_1}{f_2}$ is right convergent in x_0 and $\lim_{x_0^+} \frac{f_1}{f_2} = \frac{\lim_{x_0^+} f_1}{\lim_{x_0^+} f_2}$.

(69) Suppose f_1 is left convergent in x_0 and $\lim_{x_0^-} f_1 = 0$ and for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}(f_1 f_2)$ and there exists r such that $0 < r$ and f_2 is bounded on $]x_0 - r, x_0[$. Then $f_1 f_2$ is left convergent in x_0 and $\lim_{x_0^-}(f_1 f_2) = 0$.

(70) Suppose f_1 is right convergent in x_0 and $\lim_{x_0^+} f_1 = 0$ and for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}(f_1 f_2)$ and there exists r such that $0 < r$ and f_2 is bounded on $]x_0, x_0 + r[$. Then $f_1 f_2$ is right convergent in x_0 and $\lim_{x_0^+}(f_1 f_2) = 0$.

(71) Suppose that

(i) f_1 is left convergent in x_0,

(ii) f_2 is left convergent in x_0,

(iii) $\lim_{x_0^-} f_1 = \lim_{x_0^-} f_2$,

(iv) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \mathrm{dom}\, f$,

(v) there exists r such that $0 < r$ and for every g such that $g \in \mathrm{dom}\, f \cap]x_0 - r, x_0[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$ but $\mathrm{dom}\, f_1 \cap]x_0 - r, x_0[\subseteq \mathrm{dom}\, f_2 \cap]x_0 - r, x_0[$ and $\mathrm{dom}\, f \cap]x_0 - r, x_0[\subseteq \mathrm{dom}\, f_1 \cap]x_0 - r, x_0[$ or $\mathrm{dom}\, f_2 \cap]x_0 - r, x_0[\subseteq \mathrm{dom}\, f_1 \cap]x_0 - r, x_0[$ and $\mathrm{dom}\, f \cap]x_0 - r, x_0[\subseteq \mathrm{dom}\, f_2 \cap]x_0 - r, x_0[$.

Then f is left convergent in x_0 and $\lim_{x_0^-} f = \lim_{x_0^-} f_1$.

(72) Suppose that

(i) f_1 is left convergent in x_0,

(ii) f_2 is left convergent in x_0,

(iii) $\lim_{x_0^-} f_1 = \lim_{x_0^-} f_2$,

(iv) there exists r such that $0 < r$ and $]x_0 - r, x_0[\subseteq (\mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2) \cap \mathrm{dom}\, f$ and for every g such that $g \in]x_0 - r, x_0[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$.

Then f is left convergent in x_0 and $\lim_{x_0^-} f = \lim_{x_0^-} f_1$.

(73) Suppose that

(i) f_1 is right convergent in x_0,

(ii) f_2 is right convergent in x_0,

(iii) $\lim_{x_0^+} f_1 = \lim_{x_0^+} f_2$,

(iv) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \mathrm{dom}\, f$,

(v) there exists r such that $0 < r$ and for every g such that $g \in \mathrm{dom}\, f \cap]x_0, x_0 + r[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$ but $\mathrm{dom}\, f_1 \cap]x_0, x_0 + r[\subseteq$

$\operatorname{dom} f_2 \cap]x_0, x_0 + r[$ and $\operatorname{dom} f \cap]x_0, x_0 + r[\subseteq \operatorname{dom} f_1 \cap]x_0, x_0 + r[$ or $\operatorname{dom} f_2 \cap]x_0, x_0 + r[\subseteq \operatorname{dom} f_1 \cap]x_0, x_0 + r[$ and $\operatorname{dom} f \cap]x_0, x_0 + r[\subseteq \operatorname{dom} f_2 \cap]x_0, x_0 + r[$.

Then f is right convergent in x_0 and $\lim_{x_0^+} f = \lim_{x_0^+} f_1$.

(74) Suppose that
(i) f_1 is right convergent in x_0,
(ii) f_2 is right convergent in x_0,
(iii) $\lim_{x_0^+} f_1 = \lim_{x_0^+} f_2$,
(iv) there exists r such that $0 < r$ and $]x_0, x_0 + r[\subseteq (\operatorname{dom} f_1 \cap \operatorname{dom} f_2) \cap \operatorname{dom} f$ and for every g such that $g \in]x_0, x_0 + r[$ holds $f_1(g) \leq f(g)$ and $f(g) \leq f_2(g)$.

Then f is right convergent in x_0 and $\lim_{x_0^+} f = \lim_{x_0^+} f_1$.

(75) Suppose that
(i) f_1 is left convergent in x_0,
(ii) f_2 is left convergent in x_0,
(iii) there exists r such that $0 < r$ but $\operatorname{dom} f_1 \cap]x_0 - r, x_0[\subseteq \operatorname{dom} f_2 \cap]x_0 - r, x_0[$ and for every g such that $g \in \operatorname{dom} f_1 \cap]x_0 - r, x_0[$ holds $f_1(g) \leq f_2(g)$ or $\operatorname{dom} f_2 \cap]x_0 - r, x_0[\subseteq \operatorname{dom} f_1 \cap]x_0 - r, x_0[$ and for every g such that $g \in \operatorname{dom} f_2 \cap]x_0 - r, x_0[$ holds $f_1(g) \leq f_2(g)$.

Then $\lim_{x_0^-} f_1 \leq \lim_{x_0^-} f_2$.

(76) Suppose that
(i) f_1 is right convergent in x_0,
(ii) f_2 is right convergent in x_0,
(iii) there exists r such that $0 < r$ but $\operatorname{dom} f_1 \cap]x_0, x_0 + r[\subseteq \operatorname{dom} f_2 \cap]x_0, x_0 + r[$ and for every g such that $g \in \operatorname{dom} f_1 \cap]x_0, x_0 + r[$ holds $f_1(g) \leq f_2(g)$ or $\operatorname{dom} f_2 \cap]x_0, x_0 + r[\subseteq \operatorname{dom} f_1 \cap]x_0, x_0 + r[$ and for every g such that $g \in \operatorname{dom} f_2 \cap]x_0, x_0 + r[$ holds $f_1(g) \leq f_2(g)$.

Then $\lim_{x_0^+} f_1 \leq \lim_{x_0^+} f_2$.

(77) If f is left divergent to $+\infty$ in x_0 or f is left divergent to $-\infty$ in x_0 but for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$, then $\frac{1}{f}$ is left convergent in x_0 and $\lim_{x_0^-} \frac{1}{f} = 0$.

One can prove the following propositions:

(78) If f is right divergent to $+\infty$ in x_0 or f is right divergent to $-\infty$ in x_0 but for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$, then $\frac{1}{f}$ is right convergent in x_0 and $\lim_{x_0^+} \frac{1}{f} = 0$.

(79) If f is left convergent in x_0 and $\lim_{x_0^-} f = 0$ and there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap]x_0 - r, x_0[$ holds $0 < f(g)$, then $\frac{1}{f}$ is left divergent to $+\infty$ in x_0.

(80) If f is left convergent in x_0 and $\lim_{x_0^-} f = 0$ and there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap]x_0 - r, x_0[$ holds $f(g) < 0$, then $\frac{1}{f}$ is left divergent to $-\infty$ in x_0.

(81) If f is right convergent in x_0 and $\lim_{x_0^+} f = 0$ and there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap]x_0, x_0+r[$ holds $0 < f(g)$, then $\frac{1}{f}$ is right divergent to $+\infty$ in x_0.

(82) If f is right convergent in x_0 and $\lim_{x_0^+} f = 0$ and there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap]x_0, x_0+r[$ holds $f(g) < 0$, then $\frac{1}{f}$ is right divergent to $-\infty$ in x_0.

(83) Suppose that
(i) f is left convergent in x_0,
(ii) $\lim_{x_0^-} f = 0$,
(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$,
(iv) there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap]x_0 - r, x_0[$ holds $0 \leq f(g)$.
Then $\frac{1}{f}$ is left divergent to $+\infty$ in x_0.

(84) Suppose that
(i) f is left convergent in x_0,
(ii) $\lim_{x_0^-} f = 0$,
(iii) for every r such that $r < x_0$ there exists g such that $r < g$ and $g < x_0$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$,
(iv) there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap]x_0 - r, x_0[$ holds $f(g) \leq 0$.
Then $\frac{1}{f}$ is left divergent to $-\infty$ in x_0.

(85) Suppose that
(i) f is right convergent in x_0,
(ii) $\lim_{x_0^+} f = 0$,
(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$,
(iv) there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap]x_0, x_0 + r[$ holds $0 \leq f(g)$.
Then $\frac{1}{f}$ is right divergent to $+\infty$ in x_0.

(86) Suppose that
(i) f is right convergent in x_0,
(ii) $\lim_{x_0^+} f = 0$,
(iii) for every r such that $x_0 < r$ there exists g such that $g < r$ and $x_0 < g$ and $g \in \operatorname{dom} f$ and $f(g) \neq 0$,
(iv) there exists r such that $0 < r$ and for every g such that $g \in \operatorname{dom} f \cap]x_0, x_0 + r[$ holds $f(g) \leq 0$.
Then $\frac{1}{f}$ is right divergent to $-\infty$ in x_0.

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