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Directed Geometrical Bundles and Their Analytical Representation [1]

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Summary. We introduce the notion of weak directed geometrical bundle. We prove representation theorems for directed and weak directed geometrical bundles which establish a one-to-one correspondence between such structures and appropriate 2-divisible abelian groups. To this aim we construct over an arbitrary weak directed geometrical bundle a group defined entirely in terms of geometrical notions - the group of (abstract) "free vectors".

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The terminology and notation used here have been introduced in the following articles: [8], [3], [4], [10], [11], [7], [5], [6], [1], [9], and [2]. An affine structure is said to be a weak affine vector space if:

(Def.1) (i) there exist elements a, b of the points of it such that $a \neq b$,

(ii) for all elements a, b, c of the points of it such that $a, b \Rrightarrow c, c$ holds $a = b$,

(iii) for all elements a, b, c, d, p, q of the points of it such that $a, b \Rrightarrow p, q$ and $c, d \Rrightarrow p, q$ holds $a, b \Rrightarrow c, d$,

(iv) for every elements a, b, c of the points of it there exists an element d of the points of it such that $a, b \Rrightarrow c, d$,

(v) for all elements a, b, c, a', b', c' of the points of it such that $a, b \Rrightarrow a', b'$ and $a, c \Rrightarrow a', c'$ holds $b, c \Rrightarrow b', c'$,

(vi) for every elements a, c of the points of it there exists an element b of the points of it such that $a, b \Rrightarrow b, c$,

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(vii) for all elements a, b, c, d of the points of it such that $a, b \Rrightarrow c, d$ holds $a, c \Rrightarrow b, d$.

We see that the space of free vectors is a weak affine vector space.

We adopt the following convention: A_1 will be a weak affine vector space and a, b, c, d, f, a', b', c', d', f', p, q, r, o will be elements of the points of A_1. The following propositions are true:

(2)[2] $a, b \Rrightarrow a, b$.

(3) $a, a \Rrightarrow a, a$.

(4) If $a, b \Rrightarrow c, d$, then $c, d \Rrightarrow a, b$.

(5) If $a, b \Rrightarrow a, c$, then $b = c$.

(6) If $a, b \Rrightarrow c, d$ and $a, b \Rrightarrow c, d'$, then $d = d'$.

(7) For all a, b holds $a, a \Rrightarrow b, b$.

(8) If $a, b \Rrightarrow c, d$, then $b, a \Rrightarrow d, c$.

(9) If $a, b \Rrightarrow c, d$ and $a, c \Rrightarrow b', d$, then $b = b'$.

(10) If $b, c \Rrightarrow b', c'$ and $a, d \Rrightarrow b, c$ and $a, d' \Rrightarrow b', c'$, then $d = d'$.

(11) If $a, b \Rrightarrow a', b'$ and $c, d \Rrightarrow b, a$ and $c, d' \Rrightarrow b', a'$, then $d = d'$.

(12) If $a, b \Rrightarrow a', b'$ and $c, d \Rrightarrow c', d'$ and $b, f \Rrightarrow c, d$ and $b', f' \Rrightarrow c', d'$, then $a, f \Rrightarrow a', f'$.

(13) If $a, b \Rrightarrow a', b'$ and $a, c \Rrightarrow c', b'$, then $b, c \Rrightarrow c', a'$.

Let us consider A_1, a, b. We say that a, b are in a maximal distance if and only if:

(Def.2) $a, b \Rrightarrow b, a$ and $a \neq b$.

One can prove the following propositions:

(15)[3] a, a are not in a maximal distance.

(16) There exist a, b such that $a \neq b$ and a, b are not in a maximal distance.

(17) If a, b are in a maximal distance, then b, a are in a maximal distance.

(18) If a, b are in a maximal distance and a, c are in a maximal distance, then $b = c$ or b, c are in a maximal distance.

(19) If a, b are in a maximal distance and $a, b \Rrightarrow c, d$, then c, d are in a maximal distance.

Let us consider A_1, a, b, c. We say that b is a midpoint of a, c if and only if:

(Def.3) $a, b \Rrightarrow b, c$.

We now state a number of propositions:

(21)[4] If b is a midpoint of a, c, then b is a midpoint of c, a.

(22) b is a midpoint of a, b if and only if $a = b$.

(23) b is a midpoint of a, a if and only if $a = b$ or a, b are in a maximal distance.

[2]The proposition (1) was either repeated or obvious.
[3]The proposition (14) was either repeated or obvious.
[4]The proposition (20) was either repeated or obvious.

(24) There exists b such that b is a midpoint of a, c.

(25) If b is a midpoint of a, c and b' is a midpoint of a, c, then $b = b'$ or b, b' are in a maximal distance.

(26) There exists c such that b is a midpoint of a, c.

(27) If b is a midpoint of a, c and b is a midpoint of a, c', then $c = c'$.

(28) If b is a midpoint of a, c and b, b' are in a maximal distance, then b' is a midpoint of a, c.

(29) If b is a midpoint of a, c and b' is a midpoint of a, c' and b, b' are in a maximal distance, then $c = c'$.

(30) If p is a midpoint of a, a' and p is a midpoint of b, b', then $a, b \Rrightarrow b', a'$.

(31) If p is a midpoint of a, a' and q is a midpoint of b, b' and p, q are in a maximal distance, then $a, b \Rrightarrow b', a'$.

Let us consider A_1, a, b. The functor $\mathrm{PSym}(a, b)$ yields an element of the points of A_1 and is defined as follows:

(Def.4) a is a midpoint of b, $\mathrm{PSym}(a, b)$.

One can prove the following propositions:

(32) $\mathrm{PSym}(p, a) = b$ if and only if p is a midpoint of a, b.

(33) $\mathrm{PSym}(p, a) = b$ if and only if $a, p \Rrightarrow p, b$.

(34) p is a midpoint of a, $\mathrm{PSym}(p, a)$.

(35) $\mathrm{PSym}(p, a) = a$ if and only if $a = p$ or a, p are in a maximal distance.

(36) $\mathrm{PSym}(p, \mathrm{PSym}(p, a)) = a$.

(37) If $\mathrm{PSym}(p, a) = \mathrm{PSym}(p, b)$, then $a = b$.

(38) There exists a such that $\mathrm{PSym}(p, a) = b$.

(39) $a, b \Rrightarrow \mathrm{PSym}(p, b), \mathrm{PSym}(p, a)$.

(40) $a, b \Rrightarrow c, d$ if and only if $\mathrm{PSym}(p, a), \mathrm{PSym}(p, b) \Rrightarrow \mathrm{PSym}(p, c), \mathrm{PSym}(p, d)$.

(41) a, b are in a maximal distance if and only if $\mathrm{PSym}(p, a)$, $\mathrm{PSym}(p, b)$ are in a maximal distance.

(42) b is a midpoint of a, c if and only if $\mathrm{PSym}(p, b)$ is a midpoint of $\mathrm{PSym}(p, a)$, $\mathrm{PSym}(p, c)$.

(43) $\mathrm{PSym}(p, a) = \mathrm{PSym}(q, a)$ if and only if $p = q$ or p, q are in a maximal distance.

(44) $\mathrm{PSym}(q, \mathrm{PSym}(p, \mathrm{PSym}(q, a))) = \mathrm{PSym}(\mathrm{PSym}(q, p), a)$.

(45) $\mathrm{PSym}(p, \mathrm{PSym}(q, a)) = \mathrm{PSym}(q, \mathrm{PSym}(p, a))$ if and only if $p = q$ or p, q are in a maximal distance or q, $\mathrm{PSym}(p, q)$ are in a maximal distance.

(46) $\mathrm{PSym}(p, \mathrm{PSym}(q, \mathrm{PSym}(r, a))) = \mathrm{PSym}(r, \mathrm{PSym}(q, \mathrm{PSym}(p, a)))$.

(47) There exists d such that $\mathrm{PSym}(a, \mathrm{PSym}(b, \mathrm{PSym}(c, p))) = \mathrm{PSym}(d, p)$.

(48) There exists c such that $\mathrm{PSym}(a, \mathrm{PSym}(c, p)) = \mathrm{PSym}(c, \mathrm{PSym}(b, p))$.

Let us consider A_1, o, a, b. The functor $\mathrm{Padd}(o, a, b)$ yielding an element of the points of A_1 is defined as follows:

(Def.5) $o, a \Rrightarrow b, \mathrm{Padd}(o, a, b)$.

Next we state the proposition

(49) $\mathrm{Padd}(o, a, b) = c$ if and only if $o, a \Rrightarrow b, c$.

Let us consider A_1, o, a. The functor $\mathrm{Pcom}(o, a)$ yielding an element of the points of A_1 is defined as follows:

(Def.6) o is a midpoint of a, $\mathrm{Pcom}(o, a)$.

One can prove the following propositions:

(50) $\mathrm{Pcom}(o, a) = b$ if and only if o is a midpoint of a, b.

(51) $\mathrm{Pcom}(o, a) = b$ if and only if $a, o \Rrightarrow o, b$.

Let us consider A_1, o. The functor $\mathrm{Padd}\, o$ yielding a binary operation on the points of A_1 is defined as follows:

(Def.7) for all a, b holds $(\mathrm{Padd}\, o)(a,\ b) = \mathrm{Padd}(o, a, b)$.

Let us consider A_1, o. The functor $\mathrm{Pcom}\, o$ yielding a unary operation on the points of A_1 is defined as follows:

(Def.8) for every a holds $(\mathrm{Pcom}\, o)(a) = \mathrm{Pcom}(o, a)$.

The following propositions are true:

(52) For every binary operation O on the points of A_1 holds $O = \mathrm{Padd}\, o$ if and only if for all a, b holds $O(a,\ b) = \mathrm{Padd}(o, a, b)$.

(53) For every unary operation O on the points of A_1 holds $O = \mathrm{Pcom}\, o$ if and only if for every a holds $O(a) = \mathrm{Pcom}(o, a)$.

Let us consider A_1, o. The functor $\mathrm{GroupVect}(A_1, o)$ yields a group structure and is defined by:

(Def.9) $\mathrm{GroupVect}(A_1, o) = \langle$ the points of $A_1, \mathrm{Padd}\, o, \mathrm{Pcom}\, o, o\rangle$.

The following two propositions are true:

(54) For every X being a group structure holds $X = \mathrm{GroupVect}(A_1, o)$ if and only if $X = \langle$ the points of $A_1, \mathrm{Padd}\, o, \mathrm{Pcom}\, o, o\rangle$.

(55) For all A_1, o holds the carrier of $\mathrm{GroupVect}(A_1, o) =$ the points of A_1 and the addition of $\mathrm{GroupVect}(A_1, o) = \mathrm{Padd}\, o$ and the reverse-map of $\mathrm{GroupVect}(A_1, o) = \mathrm{Pcom}\, o$ and the zero of $\mathrm{GroupVect}(A_1, o) = o$.

In the sequel a, b, c will denote elements of $\mathrm{GroupVect}(A_1, o)$. One can prove the following propositions:

(56) For an arbitrary x holds x is an element of the points of A_1 if and only if x is an element of $\mathrm{GroupVect}(A_1, o)$.

(57) For all elements a, b of $\mathrm{GroupVect}(A_1, o)$ and for all elements a', b' of the points of A_1 such that $a = a'$ and $b = b'$ holds $a + b = (\mathrm{Padd}\, o)(a',\ b')$.

(58) For every element a of $\mathrm{GroupVect}(A_1, o)$ and for every element a' of the points of A_1 such that $a = a'$ holds $-a = (\mathrm{Pcom}\, o)(a')$.

(59) $0_{\mathrm{GroupVect}(A_1, o)} = o$.

(60) For every uniquely 2-divisible group A_2 and for all elements a, b of A_2 and for all elements a', b' of the carrier of A_2 such that $a = a'$ and $b = b'$ holds $a + b = a' \# b'$.

(61) $a + b = b + a$.

(62) $(a + b) + c = a + (b + c)$.

(63) $a + 0_{\mathrm{GroupVect}(A_1, o)} = a$.

(64) $a + (-a) = 0_{\mathrm{GroupVect}(A_1, o)}$.

(65) $\mathrm{GroupVect}(A_1, o)$ is an Abelian group.

Let us consider A_1, o. Then $\mathrm{GroupVect}(A_1, o)$ is an Abelian group.

In the sequel a, b will be elements of the carrier of $\mathrm{GroupVect}(A_1, o)$. Next we state the proposition

(66) For every a there exists b such that (the addition of $\mathrm{GroupVect}(A_1, o))(b, b) = a$.

Let us consider A_1, o. Then $\mathrm{GroupVect}(A_1, o)$ is a 2-divisible group.

In the sequel A_1 will denote a space of free vectors and o will denote an element of the points of A_1. One can prove the following proposition

(67) For every element a of the carrier of $\mathrm{GroupVect}(A_1, o)$ such that (the addition of
$\mathrm{GroupVect}(A_1, o))(a, a) = 0_{\mathrm{GroupVect}(A_1, o)}$
holds $a = 0_{\mathrm{GroupVect}(A_1, o)}$.

Let us consider A_1, o. Then $\mathrm{GroupVect}(A_1, o)$ is a uniquely 2-divisible group.

A uniquely 2-divisible group is said to be a proper uniquely two divisible group if:

(Def.10) there exist elements a, b of the carrier of it such that $a \neq b$.

The following proposition is true

(69)[5] $\mathrm{GroupVect}(A_1, o)$ is a proper uniquely two divisible group.

Let us consider A_1, o. Then $\mathrm{GroupVect}(A_1, o)$ is a proper uniquely two divisible group.

Next we state the proposition

(70) For every proper uniquely two divisible group A_2 holds $\mathrm{Vectors}(A_2)$ is a space of free vectors.

Let A_2 be a proper uniquely two divisible group. Then $\mathrm{Vectors}(A_2)$ is a space of free vectors.

We now state two propositions:

(71) For every A_1 and for every element o of the points of A_1 holds $A_1 = \mathrm{Vectors}(\mathrm{GroupVect}(A_1, o))$.

(72) For every A_3 being an affine structure holds A_3 is a space of free vectors if and only if there exists a proper uniquely two divisible group A_2 such that $A_3 = \mathrm{Vectors}(A_2)$.

[5]The proposition (68) was either repeated or obvious.

Let X, Y be group structures, and let f be a function from the carrier of X into the carrier of Y. We say that f is an isomorphism of X and Y if and only if:

(Def.11) f is one-to-one and $\mathrm{rng}\, f =$ the carrier of Y and for all elements a, b of X holds $f(a+b) = f(a) + f(b)$ and $f(0_X) = 0_Y$ and $f(-a) = -f(a)$.

Let X, Y be group structures. We say that X, Y are isomorph if and only if:

(Def.12) there exists a function f from the carrier of X into the carrier of Y such that f is an isomorphism of X and Y.

In the sequel A_2 will be a proper uniquely two divisible group and f will be a function from the carrier of A_2 into the carrier of A_2. The following propositions are true:

(75)[6] Let o' be an element of A_2. Let o be an element of the points of $\mathrm{Vectors}(A_2)$. Suppose for every element x of A_2 holds $f(x) = o' + x$ and $o = o'$. Then for all elements a, b of A_2 holds $f(a+b) = (\mathrm{Padd}\, o)(f(a), f(b))$ and $f(0_{A_2}) = 0_{\mathrm{GroupVect}(\mathrm{Vectors}(A_2),o)}$ and $f(-a) = (\mathrm{Pcom}\, o)(f(a))$.

(76) For every element o' of A_2 such that for every element b of A_2 holds $f(b) = o' + b$ holds f is one-to-one.

(77) For every element o' of A_2 and for every element o of the points of $\mathrm{Vectors}(A_2)$ such that for every element b of A_2 holds $f(b) = o' + b$ and $o = o'$ holds $\mathrm{rng}\, f =$ the carrier of $\mathrm{GroupVect}(\mathrm{Vectors}(A_2), o)$.

(78) For every proper uniquely two divisible group A_2 and for every element o' of A_2 and for every element o of the points of $\mathrm{Vectors}(A_2)$ such that $o = o'$ holds A_2, $\mathrm{GroupVect}(\mathrm{Vectors}(A_2), o)$ are isomorph.

References

[1] Grzegorz Bancerek. Curried and uncurried functions. *Formalized Mathematics*, 1(**3**):537–541, 1990.

[2] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.

[6] Grzegorz Lewandowski and Krzysztof Prażmowski. A construction of an abstract space of congruence of vectors. *Formalized Mathematics*, 1(**4**):685–688, 1990.

[7] Henryk Oryszczyszyn and Krzysztof Prażmowski. Analytical ordered affine spaces. *Formalized Mathematics*, 1(**3**):601–605, 1990.

[8] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[6]The propositions (73)–(74) were either repeated or obvious.

[9] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(**1**):97–105, 1990.

[10] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.

[11] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

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