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A Construction of Analytical Ordered Trapezium Spaces [1]

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Summary. We define, in a given real linear space, the midpoint operation on vectors and, with the help of the notions of directed parallelism of vectors and orthogonality of vectors, we define the relation of directed trapezium. We consider structures being enrichments of affine structures by a one binary operation, together with a function which assigns to every such structure its "affine" reduct. Theorems concerning midpoint operation and trapezium relation are proved, which enables us to introduce an abstract notion of (regular in fact) ordered trapezium space with midpoint, ordered trapezium space, and (unordered) trapezium space.

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The articles [11], [2], [4], [3], [13], [9], [12], [6], [7], [10], [8], [1], and [5] provide the notation and terminology for this paper. For simplicity we follow the rules: V will denote a real linear space, u, u_1, u_2, v, v_1, v_2, w, y will denote vectors of V, a, b will denote real numbers, and x, z will be arbitrary. Let us consider V, u, v, u_1, v_1. The predicate $u, v \parallel u_1, v_1$ is defined as follows:

(Def.1) $u, v \upuparrows u_1, v_1$ or $u, v \upuparrows v_1, u_1$.

The following propositions are true:

(1) If w, y span the space, then $\mathrm{OASpace}\,V$ is an ordered affine space.

(2) For all elements p, q, p_1, q_1 of the points of $\mathrm{OASpace}\,V$ such that $p = u$ and $q = v$ and $p_1 = u_1$ and $q_1 = v_1$ holds $p, q \upuparrows p_1, q_1$ if and only if $u, v \upuparrows u_1, v_1$.

(3) If w, y span the space, then for all elements p, q, p_1, q_1 of the points of $\Lambda(\mathrm{OASpace}\,V)$ such that $p = u$ and $q = v$ and $p_1 = u_1$ and $q_1 = v_1$ holds $p, q \parallel p_1, q_1$ if and only if $u, v \parallel u_1, v_1$.

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(4) If w, y span the space, then for all elements p, q, p_1, q_1 of the points of $\mathbf{AMSp}(V, w, y)$ such that $p = u$ and $q = v$ and $p_1 = u_1$ and $q_1 = v_1$ holds $p, q \parallel p_1, q_1$ if and only if $u, v \parallel u_1, v_1$.

Let us consider V, u, v. The functor $u\#v$ yielding a vector of V is defined by:

(Def.2) $u\#v + u\#v = u + v$.

One can prove the following propositions:

(5) $u\#u = u$.

(6) $u\#v = v\#u$.

(7) There exists y such that $u\#y = w$.

(8) $u\#u_1\#(v\#v_1) = u\#v\#(u_1\#v_1)$.

(9) If $u\#y = u\#w$, then $y = w$.

(10) $u, v \upuparrows y\#u, y\#v$.

(11) $u, v \parallel w\#u, w\#v$.

(12) $2 \cdot (u\#v - u) = v - u$ and $2 \cdot (v - u\#v) = v - u$.

(13) $u, u\#v \upuparrows u\#v, v$.

(14) $u, v \upuparrows u, u\#v$ and $u, v \upuparrows u\#v, v$.

(15) If $u, y \upuparrows y, v$, then $u\#y, y \upuparrows y, y\#v$.

(16) If $u, v \upuparrows u_1, v_1$, then $u, v \upuparrows u\#u_1, v\#v_1$.

Let us consider V, w, y, u, u_1, v, v_1. We say that u, u_1 and v, v_1 form a directed trapezium w.r.t. w, y if and only if:

(Def.3) $u, u_1 \upuparrows v, v_1$ and u, u_1, $u\#u_1$ and $v\#v_1$ are orthogonal w.r.t. w, y and v, v_1, $u\#u_1$ and $v\#v_1$ are orthogonal w.r.t. w, y.

We now state a number of propositions:

(17) If w, y span the space, then u, u and v, v form a directed trapezium w.r.t. w, y.

(18) If w, y span the space, then u, v and u, v form a directed trapezium w.r.t. w, y.

(19) If u, v and v, u form a directed trapezium w.r.t. w, y, then $u = v$.

(20) If w, y span the space and v_1, u and u, v_2 form a directed trapezium w.r.t. w, y, then $v_1 = u$ and $u = v_2$.

(21) If w, y span the space and u, v and u_1, v_1 form a directed trapezium w.r.t. w, y and u, v and u_2, v_2 form a directed trapezium w.r.t. w, y and $u \neq v$, then u_1, v_1 and u_2, v_2 form a directed trapezium w.r.t. w, y.

(22) If w, y span the space, then there exists a vector t of V such that u, v and u_1, t form a directed trapezium w.r.t. w, y or u, v and t, u_1 form a directed trapezium w.r.t. w, y.

(23) If w, y span the space and u, v and u_1, v_1 form a directed trapezium w.r.t. w, y, then u_1, v_1 and u, v form a directed trapezium w.r.t. w, y.

(24) If w, y span the space and u, v and u_1, v_1 form a directed trapezium w.r.t. w, y, then v, u and v_1, u_1 form a directed trapezium w.r.t. w, y.

(25) If w, y span the space and v, u_1 and v, u_2 form a directed trapezium w.r.t. w, y, then $u_1 = u_2$.

(26) If w, y span the space and u, v and u_1, v_1 form a directed trapezium w.r.t. w, y and u, v and u_1, v_2 form a directed trapezium w.r.t. w, y, then $u = v$ or $v_1 = v_2$.

(27) If w, y span the space and $u \neq u_1$ and u, u_1 and v, v_1 form a directed trapezium w.r.t. w, y but u, u_1 and v, v_2 form a directed trapezium w.r.t. w, y or u, u_1 and v_2, v form a directed trapezium w.r.t. w, y, then $v_1 = v_2$.

(28) If w, y span the space and u, v and u_1, v_1 form a directed trapezium w.r.t. w, y, then u, v and $u\#u_1$, $v\#v_1$ form a directed trapezium w.r.t. w, y.

(29) If w, y span the space and u, v and u_1, v_1 form a directed trapezium w.r.t. w, y, then u, v and $u\#v_1$, $v\#u_1$ form a directed trapezium w.r.t. w, y or u, v and $v\#u_1$, $u\#v_1$ form a directed trapezium w.r.t. w, y.

(30) Let u, u_1, u_2, v_1, v_2, t_1, t_2, w_1, w_2 be vectors of V. Then if w, y span the space and $u = u_1\#t_1$ and $u = u_2\#t_2$ and $u = v_1\#w_1$ and $u = v_2\#w_2$ and u_1, u_2 and v_1, v_2 form a directed trapezium w.r.t. w, y, then t_1, t_2 and w_1, w_2 form a directed trapezium w.r.t. w, y.

Let us consider V, w, y, u. Let us assume that w, y span the space. The functor $\pi^1_{w,y}(u)$ yielding a real number is defined as follows:

(Def.4) there exists b such that $u = \pi^1_{w,y}(u) \cdot w + b \cdot y$.

Let us consider V, w, y, u. Let us assume that w, y span the space. The functor $\pi^2_{w,y}(u)$ yields a real number and is defined by:

(Def.5) there exists a such that $u = a \cdot w + \pi^2_{w,y}(u) \cdot y$.

Let us consider V, w, y, u, v. Let us assume that w, y span the space. The functor $u \cdot_{w,y} v$ yields a real number and is defined as follows:

(Def.6) $u \cdot_{w,y} v = \pi^1_{w,y}(u) \cdot \pi^1_{w,y}(v) + \pi^2_{w,y}(u) \cdot \pi^2_{w,y}(v)$.

We now state a number of propositions:

(31) If w, y span the space, then for all u, v holds $u \cdot_{w,y} v = v \cdot_{w,y} u$.

(32) Suppose w, y span the space. Given u, v, v_1. Then
- (i) $u \cdot_{w,y} (v + v_1) = u \cdot_{w,y} v + u \cdot_{w,y} v_1$,
- (ii) $u \cdot_{w,y} (v - v_1) = u \cdot_{w,y} v - u \cdot_{w,y} v_1$,
- (iii) $(v - v_1) \cdot_{w,y} u = v \cdot_{w,y} u - v_1 \cdot_{w,y} u$,
- (iv) $(v + v_1) \cdot_{w,y} u = v \cdot_{w,y} u + v_1 \cdot_{w,y} u$.

(33) Suppose w, y span the space. Let u, v be vectors of V. Let a be a real number. Then
- (i) $(a \cdot u) \cdot_{w,y} v = a \cdot u \cdot_{w,y} v$,
- (ii) $u \cdot_{w,y} (a \cdot v) = a \cdot u \cdot_{w,y} v$,
- (iii) $(a \cdot u) \cdot_{w,y} v = u \cdot_{w,y} v \cdot a$,

(iv) $u \cdot_{w,y} (a \cdot v) = u \cdot_{w,y} v \cdot a$.

(34) If w, y span the space, then for all vectors u, v of V holds u, v are orthogonal w.r.t. w, y if and only if $u \cdot_{w,y} v = 0$.

(35) If w, y span the space, then for all vectors u, v, u_1, v_1 of V holds u, v, u_1 and v_1 are orthogonal w.r.t. w, y if and only if $(v-u) \cdot_{w,y} (v_1 - u_1) = 0$.

(36) If w, y span the space, then for all vectors u, v, v_1 of V holds $2 \cdot u \cdot_{w,y} (v \# v_1) = u \cdot_{w,y} v + u \cdot_{w,y} v_1$.

(37) If w, y span the space, then for all vectors u, v of V such that $u \neq v$ holds $(u - v) \cdot_{w,y} (u - v) \neq 0$.

(38) Suppose w, y span the space. Let p, q, u, v, v' be vectors of V. Let A be a real number. Suppose that

(i) p, q and u, v form a directed trapezium w.r.t. w, y,

(ii) $p \neq q$,

(iii) $A = ((p - q) \cdot_{w,y} (p + q) - 2 \cdot (p - q) \cdot_{w,y} u) \cdot (p - q) \cdot_{w,y} (p - q)^{-1}$,

(iv) $v' = u + A \cdot (p - q)$.

Then $v = v'$.

(39) Suppose w, y span the space. Let u, u', u_1, u_2, v_1, v_2, t_1, t_2, w_1, w_2 be vectors of V. Then if $u \neq u'$ and u, u' and u_1, t_1 form a directed trapezium w.r.t. w, y and u, u' and u_2, t_2 form a directed trapezium w.r.t. w, y and u, u' and v_1, w_1 form a directed trapezium w.r.t. w, y and u, u' and v_2, w_2 form a directed trapezium w.r.t. w, y and $u_1, u_2 \upuparrows v_1, v_2$, then $t_1, t_2 \upuparrows w_1, w_2$.

(40) Suppose w, y span the space. Then for all vectors u, u', u_1, u_2, v_1, t_1, t_2, w_1 of V such that $u \neq u'$ and u, u' and u_1, t_1 form a directed trapezium w.r.t. w, y and u, u' and u_2, t_2 form a directed trapezium w.r.t. w, y and u, u' and v_1, w_1 form a directed trapezium w.r.t. w, y and $v_1 = u_1 \# u_2$ holds $w_1 = t_1 \# t_2$.

(41) If w, y span the space, then for all vectors u, u', u_1, u_2, t_1, t_2 of V such that $u \neq u'$ and u, u' and u_1, t_1 form a directed trapezium w.r.t. w, y and u, u' and u_2, t_2 form a directed trapezium w.r.t. w, y holds u, u' and $u_1 \# u_2$, $t_1 \# t_2$ form a directed trapezium w.r.t. w, y.

(42) Suppose w, y span the space. Let u, u', u_1, u_2, v_1, v_2, t_1, t_2, w_1, w_2 be vectors of V. Suppose $u \neq u'$ and u, u' and u_1, t_1 form a directed trapezium w.r.t. w, y and u, u' and u_2, t_2 form a directed trapezium w.r.t. w, y and u, u' and v_1, w_1 form a directed trapezium w.r.t. w, y and u, u' and v_2, w_2 form a directed trapezium w.r.t. w, y and u_1, u_2, v_1 and v_2 are orthogonal w.r.t. w, y. Then t_1, t_2, w_1 and w_2 are orthogonal w.r.t. w, y.

(43) Let u, u', u_1, u_2, v_1, v_2, t_1, t_2, w_1, w_2 be vectors of V. Suppose w, y span the space and $u \neq u'$ and u, u' and u_1, t_1 form a directed trapezium w.r.t. w, y and u, u' and u_2, t_2 form a directed trapezium w.r.t. w, y and u, u' and v_1, w_1 form a directed trapezium w.r.t. w, y and u, u' and v_2, w_2 form a directed trapezium w.r.t. w, y and u_1, u_2 and v_1, v_2 form

a directed trapezium w.r.t. w, y. Then t_1, t_2 and w_1, w_2 form a directed trapezium w.r.t. w, y.

Let us consider V, w, y. The directed trapezium relation defined over V in the basis w, y yielding a binary relation on [: the vectors of V, the vectors of V :] is defined as follows:

(Def.7) $\langle x, z\rangle \in$ the directed trapezium relation defined over V in the basis w, y if and only if there exist u, u_1, v, v_1 such that $x = \langle u, u_1\rangle$ and $z = \langle v, v_1\rangle$ and u, u_1 and v, v_1 form a directed trapezium w.r.t. w, y.

The following proposition is true

(44) If w, y span the space, then $\langle\langle u, v\rangle, \langle u_1, v_1\rangle\rangle \in$ the directed trapezium relation defined over V in the basis w, y if and only if u, v and u_1, v_1 form a directed trapezium w.r.t. w, y.

Let us consider V. The midpoint operation in V yields a binary operation on the vectors of V and is defined as follows:

(Def.8) for all u, v holds (the midpoint operation in V)$(u, v) = u \# v$.

We consider affine midpoint structures which are systems ⟨points, a midpoint operation, a congruence⟩, where the points constitute a non-empty set, the midpoint operation is a binary operation on the points, and the congruence is a binary relation on [: the points, the points :].

Let us consider V, w, y. Let us assume that w, y span the space. The directed trapezium space defined over V in the basis w, y yielding a affine midpoint structure is defined as follows:

(Def.9) the directed trapezium space defined over V in the basis $w, y =$ ⟨the vectors of V, the midpoint operation in V, the directed trapezium relation defined over V in the basis w, y⟩.

The following proposition is true

(45) For all V, w, y such that w, y span the space holds the directed trapezium space defined over V in the basis $w, y =$ ⟨the vectors of V, the midpoint operation in V, the directed trapezium relation defined over V in the basis w, y⟩.

Let A_1 be a affine midpoint structure. The affine reduct of A_1 yielding an affine structure is defined by:

(Def.10) the affine reduct of $A_1 =$ ⟨ the points of A_1, the congruence of A_1⟩.

Let A_1 be a affine midpoint structure, and let a, b, c, d be elements of the points of A_1. The predicate $a, b \top^{>} c, d$ is defined by:

(Def.11) $\langle\langle a, b\rangle, \langle c, d\rangle\rangle \in$ the congruence of A_1.

Let A_1 be a affine midpoint structure, and let a, b be elements of the points of A_1. The functor $a \# b$ yielding an element of the points of A_1 is defined by:

(Def.12) $a\#b =$ (the midpoint operation of A_1)(a, b).

In the sequel a, b, a_1, b_1 denote elements of the points of the directed trapezium space defined over V in the basis w, y.

We now state three propositions:

(46) If w, y span the space, then for an arbitrary x holds x is an element of the points of the directed trapezium space defined over V in the basis w, y if and only if x is a vector of V.

(47) If w, y span the space and $u = a$ and $v = b$ and $u_1 = a_1$ and $v_1 = b_1$, then $a, b \top^{>} a_1, b_1$ if and only if u, v and u_1, v_1 form a directed trapezium w.r.t. w, y.

(48) If w, y span the space and $u = a$ and $v = b$, then $u\#v = a\#b$.

A affine midpoint structure is called an ordered midpoint trapezium space if it satisfies the condition (Def.13).

(Def.13) Let a, b, c, d, a_1, b_1, c_1, d_1, p, q be elements of the points of it . Then

(i) $a\#b = b\#a$,

(ii) $a\#a = a$,

(iii) $a\#b\#(c\#d) = a\#c\#(b\#d)$,

(iv) there exists an element p of the points of it such that $p\#a = b$,

(v) if $a\#b = a\#c$, then $b = c$,

(vi) if $a, b \top^{>} c, d$, then $a, b \top^{>} a\#c, b\#d$,

(vii) if $a, b \top^{>} c, d$, then $a, b \top^{>} a\#d, b\#c$ or $a, b \top^{>} b\#c, a\#d$,

(viii) if $a, b \top^{>} c, d$ and $a\#a_1 = p$ and $b\#b_1 = p$ and $c\#c_1 = p$ and $d\#d_1 = p$, then $a_1, b_1 \top^{>} c_1, d_1$,

(ix) if $p \neq q$ and $p, q \top^{>} a, a_1$ and $p, q \top^{>} b, b_1$ and $p, q \top^{>} c, c_1$ and $p, q \top^{>} d, d_1$ and $a, b \top^{>} c, d$, then $a_1, b_1 \top^{>} c_1, d_1$,

(x) if $a, b \top^{>} b, c$, then $a = b$ and $b = c$,

(xi) if $a, b \top^{>} a_1, b_1$ and $a, b \top^{>} c_1, d_1$ and $a \neq b$, then $a_1, b_1 \top^{>} c_1, d_1$,

(xii) if $a, b \top^{>} c, d$, then $c, d \top^{>} a, b$ and $b, a \top^{>} d, c$,

(xiii) there exists an element d of the points of it such that $a, b \top^{>} c, d$ or $a, b \top^{>} d, c$,

(xiv) if $a, b \top^{>} c, p$ and $a, b \top^{>} c, q$, then $a = b$ or $p = q$.

One can prove the following proposition

(49) If w, y span the space, then the directed trapezium space defined over V in the basis w, y is an ordered midpoint trapezium space.

An affine structure is called an ordered trapezium space if it satisfies the condition (Def.14).

(Def.14) Let a, b, c, d, a_1, b_1, c_1, d_1, p, q be elements of the points of it . Then

(i) if $a, b \parallel b, c$, then $a = b$ and $b = c$,

(ii) if $a, b \parallel a_1, b_1$ and $a, b \parallel c_1, d_1$ and $a \neq b$, then $a_1, b_1 \parallel c_1, d_1$,

(iii) if $a, b \parallel c, d$, then $c, d \parallel a, b$ and $b, a \parallel d, c$,

(iv) there exists an element d of the points of it such that $a, b \parallel c, d$ or $a, b \parallel d, c$,

(v) if $a, b \parallel c, p$ and $a, b \parallel c, q$, then $a = b$ or $p = q$.

Let M_1 be an ordered midpoint trapezium space. Then the affine reduct of M_1 is an ordered trapezium space.

We follow a convention: O_1 denotes an ordered trapezium space, a, b, c, d denote elements of the points of O_1, and a', b', c', d' denote elements of the points of $\Lambda(O_1)$. We now state two propositions:

(50) For an arbitrary x holds x is an element of the points of O_1 if and only if x is an element of the points of $\Lambda(O_1)$.

(51) If $a = a'$ and $b = b'$ and $c = c'$ and $d = d'$, then $a', b' \parallel c', d'$ if and only if $a, b \parallel c, d$ or $a, b \parallel d, c$.

An affine structure is called a trapezium space if it satisfies the condition (Def.15).

(Def.15) Let a', b', c', d', p', q' be elements of the points of it . Then
  (i) $a', b' \parallel b', a'$,
  (ii) if $a', b' \parallel c', d'$ and $a', b' \parallel c', q'$, then $a' = b'$ or $d' = q'$,
  (iii) if $p' \neq q'$ and $p', q' \parallel a', b'$ and $p', q' \parallel c', d'$, then $a', b' \parallel c', d'$,
  (iv) if $a', b' \parallel c', d'$, then $c', d' \parallel a', b'$,
  (v) there exists an element x' of the points of it such that $a', b' \parallel c', x'$.

Let O_1 be an ordered trapezium space. Then $\Lambda(O_1)$ is a trapezium space.

An affine structure is regular if it satisfies the condition (Def.16).

(Def.16) Let p, q, a, a_1, b, b_1, c, c_1, d, d_1 be elements of the points of it . Then if $p \neq q$ and $p, q \parallel a, a_1$ and $p, q \parallel b, b_1$ and $p, q \parallel c, c_1$ and $p, q \parallel d, d_1$ and $a, b \parallel c, d$, then $a_1, b_1 \parallel c_1, d_1$.

Let M_1 be an ordered midpoint trapezium space. Then the affine reduct of M_1 is an regular ordered trapezium space.

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On Projections in Projective Planes. Part II [1]

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Summary. We study in greater datail projectivities on Desarguesian projective planes. We are particularly interested in the situation when the composition of given two projectivities can be replaced by another two, with a given axis or centre of one of them.

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The articles [7], [9], [6], [8], [10], [11], [5], [4], [1], [2], and [3] provide the notation and terminology for this paper. In the sequel I_1 will denote a projective space defined in terms of incidence and z will denote an element of the points of I_1. Let us consider I_1, and let A, B, C be elements of the lines of I_1. We say that A, B, C are concurrent if and only if:

(Def.1) there exists an element o of the points of I_1 such that $o \mid A$ and $o \mid B$ and $o \mid C$.

Let us consider I_1, and let Z be an element of the lines of I_1. The functor chain(Z) yields a subset of the points of I_1 and is defined by:

(Def.2) $\mathrm{chain}(Z) = \{z : z \mid Z\}$.

We adopt the following rules: I_2 will denote an Desarguesian 2-dimensional projective space defined in terms of incidence, a, b, c, d, p, p'_1, q, o, o', o'', o'_1, r, s, x, y, o_1, o_2 will denote elements of the points of I_2, and O_1, O_2, O_3, A, B, C, O, Q, R, S will denote elements of the lines of I_2. Let us consider I_2. A partial function from the points of I_2 to the points of I_2 is said to be a projection of I_2 if:

(Def.3) there exist a, A, B such that $a \nmid A$ and $a \nmid B$ and it $= \pi_a(A \to B)$.

The following propositions are true:

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(1) If $A = B$ or $B = C$ or $C = A$, then A, B, C are concurrent.

(2) If A, B, C are concurrent, then A, C, B are concurrent and B, A, C are concurrent and B, C, A are concurrent and C, A, B are concurrent and C, B, A are concurrent.

(3) If $o \nmid A$ and $o \nmid B$ and $y \mid B$, then there exists x such that $x \mid A$ and $\pi_o(A \to B)(x) = y$.

(4) If $o \nmid A$ and $o \nmid B$, then $\operatorname{rng} \pi_o(A \to B) \subseteq$ the points of I_2.

(5) If $o \nmid A$ and $o \nmid B$, then $\operatorname{dom} \pi_o(A \to B) = \operatorname{chain}(A)$.

(6) If $o \nmid A$ and $o \nmid B$, then $\operatorname{rng} \pi_o(A \to B) = \operatorname{chain}(B)$.

(7) For an arbitrary x holds $x \in \operatorname{chain}(A)$ if and only if there exists a such that $x = a$ and $a \mid A$.

(8) If $o \nmid A$ and $o \nmid B$, then $\pi_o(A \to B)$ is one-to-one.

(9) If $o \nmid A$ and $o \nmid B$, then $\pi_o(A \to B)^{-1} = \pi_o(B \to A)$.

(10) For every projection f of I_2 holds f^{-1} is a projection of I_2.

(11) If $o \nmid A$, then $\pi_o(A \to A) = \mathrm{id}_{\operatorname{chain}(A)}$.

(12) $\mathrm{id}_{\operatorname{chain}(A)}$ is a projection of I_2.

(13) If $o \nmid A$ and $o \nmid B$ and $o \nmid C$, then $\pi_o(C \to B) \cdot \pi_o(A \to C) = \pi_o(A \to B)$.

(14) Suppose $o_1 \nmid O_1$ and $o_1 \nmid O_2$ and $o_2 \nmid O_2$ and $o_2 \nmid O_3$ and O_1, O_2, O_3 are concurrent and $O_1 \neq O_3$. Then there exists o such that $o \nmid O_1$ and $o \nmid O_3$ and $\pi_{o_2}(O_2 \to O_3) \cdot \pi_{o_1}(O_1 \to O_2) = \pi_o(O_1 \to O_3)$.

(15) Suppose that

(i) $a \nmid A$,

(ii) $b \nmid B$,

(iii) $a \nmid C$,

(iv) $b \nmid C$,

(v) A, B, C are not concurrent,

(vi) $c \mid A$,

(vii) $c \mid C$,

(viii) $c \mid Q$,

(ix) $b \nmid Q$,

(x) $A \neq Q$,

(xi) $a \neq b$,

(xii) $b \neq q$,

(xiii) $a \mid O$,

(xiv) $b \mid O$,

(xv) B, C, O are not concurrent,

(xvi) $d \mid C$,

(xvii) $d \mid B$,

(xviii) $a \mid O_1$,

(xix) $d \mid O_1$,

(xx) $p \mid A$,

(xxi) $p \mid O_1$,

(xxii) $q \mid O$,
(xxiii) $q \mid O_2$,
(xxiv) $p \mid O_2$,
(xxv) $p'_1 \mid O_2$,
(xxvi) $d \mid O_3$,
(xxvii) $b \mid O_3$,
(xxviii) $p'_1 \mid O_3$,
(xxix) $p'_1 \mid Q$,
(xxx) $Q \neq C$,
(xxxi) $q \neq a$,
(xxxii) $q \nmid A$,
(xxxiii) $q \nmid Q$.

Then $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_b(Q \to B) \cdot \pi_q(A \to Q)$.

(16) Suppose that
(i) $a \nmid A$,
(ii) $a \nmid C$,
(iii) $b \nmid B$,
(iv) $b \nmid C$,
(v) $b \nmid Q$,
(vi) A, B, C are not concurrent,
(vii) $a \neq b$,
(viii) $b \neq q$,
(ix) $A \neq Q$,
(x) $c, o \mid A$,
(xi) $o, o'', d \mid B$,
(xii) $c, d, o' \mid C$,
(xiii) $a, b, d \mid O$,
(xiv) $c, o'_1 \mid Q$,
(xv) $a, o, o' \mid O_1$,
(xvi) $b, o', o'_1 \mid O_2$,
(xvii) $o, o'_1, q \mid O_3$,
(xviii) $q \mid O$.

Then $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_b(Q \to B) \cdot \pi_q(A \to Q)$.

(17) Suppose that
(i) $a \nmid A$,
(ii) $a \nmid C$,
(iii) $b \nmid B$,
(iv) $b \nmid C$,
(v) $b \nmid Q$,
(vi) A, B, C are not concurrent,
(vii) B, C, O are not concurrent,
(viii) $A \neq Q$,
(ix) $Q \neq C$,
(x) $a \neq b$,

(xi) $c, p \mid A$,
(xii) $d \mid B$,
(xiii) $c, d \mid C$,
(xiv) $a, b, q \mid O$,
(xv) $c, p'_1 \mid Q$,
(xvi) $a, d, p \mid O_1$,
(xvii) $q, p, p'_1 \mid O_2$,
(xviii) $b, d, p'_1 \mid O_3$.

Then $q \neq a$ and $q \neq b$ and $q \nmid A$ and $q \nmid Q$.

(18) Suppose that
(i) $a \nmid A$,
(ii) $a \nmid C$,
(iii) $b \nmid B$,
(iv) $b \nmid C$,
(v) $b \nmid Q$,
(vi) A, B, C are not concurrent,
(vii) $a \neq b$,
(viii) $A \neq Q$,
(ix) $c, o \mid A$,
(x) $o, o'', d \mid B$,
(xi) $c, d, o' \mid C$,
(xii) $a, b, d \mid O$,
(xiii) $c, o'_1 \mid Q$,
(xiv) $a, o, o' \mid O_1$,
(xv) $b, o', o'_1 \mid O_2$,
(xvi) $o, o'_1, q \mid O_3$,
(xvii) $q \mid O$.

Then $q \nmid A$ and $q \nmid Q$ and $b \neq q$.

(19) Suppose that
(i) $a \nmid A$,
(ii) $a \nmid C$,
(iii) $b \nmid B$,
(iv) $b \nmid C$,
(v) $q \nmid A$,
(vi) A, B, C are not concurrent,
(vii) B, C, O are not concurrent,
(viii) $a \neq b$,
(ix) $b \neq q$,
(x) $q \neq a$,
(xi) $c, p \mid A$,
(xii) $d \mid B$,
(xiii) $c, d \mid C$,
(xiv) $a, b, q \mid O$,
(xv) $c, p'_1 \mid Q$,

(xvi) $a, d, p \mid O_1$,
(xvii) $q, p, p'_1 \mid O_2$,
(xviii) $b, d, p'_1 \mid O_3$.

Then $Q \neq A$ and $Q \neq C$ and $q \nmid Q$ and $b \nmid Q$.

(20) Suppose that

(i) $a \nmid A$,
(ii) $a \nmid C$,
(iii) $b \nmid B$,
(iv) $b \nmid C$,
(v) $q \nmid A$,
(vi) A, B, C are not concurrent,
(vii) $a \neq b$,
(viii) $b \neq q$,
(ix) $c, o \mid A$,
(x) $o, o'', d \mid B$,
(xi) $c, d, o' \mid C$,
(xii) $a, b, d \mid O$,
(xiii) $c, o'_1 \mid Q$,
(xiv) $a, o, o' \mid O_1$,
(xv) $b, o', o'_1 \mid O_2$,
(xvi) $o, o'_1, q \mid O_3$,
(xvii) $q \mid O$.

Then $b \nmid Q$ and $q \nmid Q$ and $A \neq Q$.

(21) Suppose that

(i) $a \nmid A$,
(ii) $b \nmid B$,
(iii) $a \nmid C$,
(iv) $b \nmid C$,
(v) A, B, C are not concurrent,
(vi) A, C, Q are concurrent,
(vii) $b \nmid Q$,
(viii) $A \neq Q$,
(ix) $a \neq b$,
(x) $a \mid O$,
(xi) $b \mid O$.

Then there exists q such that $q \mid O$ and $q \nmid A$ and $q \nmid Q$ and $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_b(Q \to B) \cdot \pi_q(A \to Q)$.

(22) Suppose that

(i) $a \nmid A$,
(ii) $b \nmid B$,
(iii) $a \nmid C$,
(iv) $b \nmid C$,
(v) A, B, C are not concurrent,
(vi) B, C, Q are concurrent,

(vii) $a \nmid Q$,
(viii) $B \neq Q$,
(ix) $a \neq b$,
(x) $a \mid O$,
(xi) $b \mid O$.

Then there exists q such that $q \mid O$ and $q \nmid B$ and $q \nmid Q$ and $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_q(Q \to B) \cdot \pi_a(A \to Q)$.

(23) Suppose that
(i) $a \nmid A$,
(ii) $b \nmid B$,
(iii) $a \nmid C$,
(iv) $b \nmid C$,
(v) $a \nmid B$,
(vi) $b \nmid A$,
(vii) $c \mid A$,
(viii) $c \mid C$,
(ix) $d \mid B$,
(x) $d \mid C$,
(xi) $a \mid S$,
(xii) $d \mid S$,
(xiii) $c \mid R$,
(xiv) $b \mid R$,
(xv) $s \mid A$,
(xvi) $s \mid S$,
(xvii) $r \mid B$,
(xviii) $r \mid R$,
(xix) $s \mid Q$,
(xx) $r \mid Q$,
(xxi) A, B, C are not concurrent.

Then $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_a(Q \to B) \cdot \pi_b(A \to Q)$.

(24) Suppose $a \nmid A$ and $b \nmid B$ and $a \nmid C$ and $b \nmid C$ and $a \neq b$ and $a \mid O$ and $b \mid O$ and $q \mid O$ and $q \nmid A$ and $q \neq b$ and A, B, C are not concurrent. Then there exists Q such that A, C, Q are concurrent and $b \nmid Q$ and $q \nmid Q$ and $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_b(Q \to B) \cdot \pi_q(A \to Q)$.

(25) Suppose $a \nmid A$ and $b \nmid B$ and $a \nmid C$ and $b \nmid C$ and $a \neq b$ and $a \mid O$ and $b \mid O$ and $q \mid O$ and $q \nmid B$ and $q \neq a$ and A, B, C are not concurrent. Then there exists Q such that B, C, Q are concurrent and $a \nmid Q$ and $q \nmid Q$ and $\pi_b(C \to B) \cdot \pi_a(A \to C) = \pi_q(Q \to B) \cdot \pi_a(A \to Q)$.

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Metric-Affine Configurations in Metric Affine Planes - Part I

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Summary. We introduce several configurational axioms for metric affine planes such as theorem on three perpendiculars, orthogonalization of major Desargues Axiom, orthogonalization of the trapezium variant of Desargues Axiom, axiom on parallel projection together with its indirect forms. For convenience we also consider affine Major Desargues Axiom. The aim is to prove logical relationships which hold between the introduced statements.

MML Identifier: CONAFFM.

The notation and terminology used here have been introduced in the following papers: [7], [8], [6], [3], [5], [4], [1], and [2]. We adopt the following rules: X will denote a metric affine plane and o, a, a_1, b, b_1, c, c_1 will denote elements of the points of X. Let us consider X. We say that Desargues Axiom holds in X if and only if the condition (Def.1) is satisfied.

(Def.1) Given o, a, a_1, b, b_1, c, c_1. Suppose that
- (i) $o \neq a$,
- (ii) $o \neq a_1$,
- (iii) $o \neq b$,
- (iv) $o \neq b_1$,
- (v) $o \neq c$,
- (vi) $o \neq c_1$,
- (vii) not $\mathbf{L}(b, b_1, a)$,
- (viii) not $\mathbf{L}(a, a_1, c)$,
- (ix) $\mathbf{L}(o, a, a_1)$,
- (x) $\mathbf{L}(o, b, b_1)$,
- (xi) $\mathbf{L}(o, c, c_1)$,
- (xii) $a, b \parallel a_1, b_1$,
- (xiii) $a, c \parallel a_1, c_1$.

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Then $b, c \parallel b_1, c_1$.

Let us consider X. We say that AH holds in X if and only if the condition (Def.2) is satisfied.

(Def.2) Given o, a, a_1, b, b_1, c, c_1. Suppose $o, a \perp o, a_1$ and $o, b \perp o, b_1$ and $o, c \perp o, c_1$ and $a, b \perp a_1, b_1$ and $o, a \parallel b, c$ and $a, c \perp a_1, c_1$ and $o, c \nparallel o, a$ and $o, a \nparallel o, b$. Then $b, c \perp b_1, c_1$.

Let us consider X. We say that theorem on three perpendiculars holds in X if and only if:

(Def.3) for all a, b, c such that not $\mathbf{L}(a, b, c)$ there exists an element d of the points of X such that $d, a \perp b, c$ and $d, b \perp a, c$ and $d, c \perp a, b$.

Let us consider X. We say that othogonal verion of Desargues Axiom holds in X if and only if the condition (Def.4) is satisfied.

(Def.4) Given o, a, a_1, b, b_1, c, c_1. Then if $o, a \perp o, a_1$ and $o, b \perp o, b_1$ and $o, c \perp o, c_1$ and $a, b \perp a_1, b_1$ and $a, c \perp a_1, c_1$ and $o, c \nparallel o, a$ and $o, a \nparallel o, b$, then $b, c \perp b_1, c_1$.

Let us consider X. We say that LIN holds in X if and only if the condition (Def.5) is satisfied.

(Def.5) Given o, a, a_1, b, b_1, c, c_1. Suppose that

(i) $o \neq a$,
(ii) $o \neq a_1$,
(iii) $o \neq b$,
(iv) $o \neq b_1$,
(v) $o \neq c$,
(vi) $o \neq c_1$,
(vii) $a \neq b$,
(viii) $o, c \perp o, c_1$,
(ix) $o, a \perp o, a_1$,
(x) $o, b \perp o, b_1$,
(xi) not $\mathbf{L}(o, c, a)$,
(xii) $\mathbf{L}(o, a, b)$,
(xiii) $\mathbf{L}(o, a_1, b_1)$,
(xiv) $a, c \perp a_1, c_1$,
(xv) $b, c \perp b_1, c_1$.

Then $a, a_1 \parallel b, b_1$.

Let us consider X. We say that first indirect form of LIN holds in X if and only if the condition (Def.6) is satisfied.

(Def.6) Given o, a, a_1, b, b_1, c, c_1. Suppose that

(i) $o \neq a$,
(ii) $o \neq a_1$,
(iii) $o \neq b$,
(iv) $o \neq b_1$,
(v) $o \neq c$,

(vi) $o \neq c_1$,
(vii) $a \neq b$,
(viii) $o, c \perp o, c_1$,
(ix) $o, a \perp o, a_1$,
(x) $o, b \perp o, b_1$,
(xi) not $\mathbf{L}(o, c, a)$,
(xii) $\mathbf{L}(o, a, b)$,
(xiii) $\mathbf{L}(o, a_1, b_1)$,
(xiv) $a, c \perp a_1, c_1$,
(xv) $a, a_1 \parallel b, b_1$.
Then $b, c \perp b_1, c_1$.

Let us consider X. We say that second indirect form of LIN holds in X if and only if the condition (Def.7) is satisfied.

(Def.7) Given o, a, a_1, b, b_1, c, c_1. Suppose that
(i) $o \neq a$,
(ii) $o \neq a_1$,
(iii) $o \neq b$,
(iv) $o \neq b_1$,
(v) $o \neq c$,
(vi) $o \neq c_1$,
(vii) $a \neq b$,
(viii) $a, a_1 \parallel b, b_1$,
(ix) $o, a \perp o, a_1$,
(x) $o, b \perp o, b_1$,
(xi) not $\mathbf{L}(o, c, a)$,
(xii) $\mathbf{L}(o, a, b)$,
(xiii) $\mathbf{L}(o, a_1, b_1)$,
(xiv) $a, c \perp a_1, c_1$,
(xv) $b, c \perp b_1, c_1$.
Then $o, c \perp o, c_1$.

We now state several propositions:

(1) If othogonal verion of Desargues Axiom holds in X, then Desargues Axiom holds in X.

(2) If othogonal verion of Desargues Axiom holds in X, then AH holds in X.

(3) If LIN holds in X, then first indirect form of LIN holds in X.

(4) If first indirect form of LIN holds in X, then second indirect form of LIN holds in X.

(5) If LIN holds in X, then othogonal verion of Desargues Axiom holds in X.

(6) If LIN holds in X, then theorem on three perpendiculars holds in X.

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Metric-Affine Configurations in Metric Affine Planes - Part II

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Summary. A continuation of [5]. We introduce more configurational axioms i.e. orthogonalizations of "scherungssatzes" (direct and indirect), "Scherungssatz" with orthogonal axes, Pappus axiom with orthogonal axes; we also consider the affine Major Pappus Axiom and affine minor Desargues Axiom. We prove a number of implications which hold between the above axioms.

MML Identifier: CONMETR.

The articles [2], [4], [1], [3], and [5] provide the notation and terminology for this paper. We adopt the following rules: X will denote a metric affine plane, o, a, a_1, a_2, a_3, a_4, b, b_1, b_2, b_3, b_4, c, c_1, d will denote elements of the points of X, and A, K, M, N will denote subsets of the points of X. Let us consider X. We say that Pappos Axiom with orthogonal axes holds in X if and only if the condition (Def.1) is satisfied.

(Def.1) Given o, a_1, a_2, a_3, b_1, b_2, b_3, M, N. Suppose that
- (i) $o \in M$,
- (ii) $a_1 \in M$,
- (iii) $a_2 \in M$,
- (iv) $a_3 \in M$,
- (v) $o \in N$,
- (vi) $b_1 \in N$,
- (vii) $b_2 \in N$,
- (viii) $b_3 \in N$,
- (ix) $b_2 \notin M$,
- (x) $a_3 \notin N$,
- (xi) $M \perp N$,
- (xii) $o \neq a_1$,

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(xiii) $o \neq a_2$,
(xiv) $o \neq a_3$,
(xv) $o \neq b_1$,
(xvi) $o \neq b_2$,
(xvii) $o \neq b_3$,
(xviii) $a_3, b_2 \parallel a_2, b_1$,
(xix) $a_3, b_3 \parallel a_1, b_1$.
Then $a_1, b_2 \parallel a_2, b_3$.

Let us consider X. We say that Pappos Axiom holds in X if and only if the condition (Def.2) is satisfied.

(Def.2) Given o, a_1, a_2, a_3, b_1, b_2, b_3, M, N. Suppose that
(i) M is a line,
(ii) N is a line,
(iii) $o \in M$,
(iv) $a_1 \in M$,
(v) $a_2 \in M$,
(vi) $a_3 \in M$,
(vii) $o \in N$,
(viii) $b_1 \in N$,
(ix) $b_2 \in N$,
(x) $b_3 \in N$,
(xi) $b_2 \notin M$,
(xii) $a_3 \notin N$,
(xiii) $o \neq a_1$,
(xiv) $o \neq a_2$,
(xv) $o \neq a_3$,
(xvi) $o \neq b_1$,
(xvii) $o \neq b_2$,
(xviii) $o \neq b_3$,
(xix) $a_3, b_2 \parallel a_2, b_1$,
(xx) $a_3, b_3 \parallel a_1, b_1$.
Then $a_1, b_2 \parallel a_2, b_3$.

Let us consider X. We say that MH1 holds in X if and only if the condition (Def.3) is satisfied.

(Def.3) Given a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, M, N. Suppose that
(i) $M \perp N$,
(ii) $a_1 \in M$,
(iii) $a_3 \in M$,
(iv) $b_1 \in M$,
(v) $b_3 \in M$,
(vi) $a_2 \in N$,
(vii) $a_4 \in N$,
(viii) $b_2 \in N$,
(ix) $b_4 \in N$,

(x) $a_2 \notin M$,
(xi) $a_4 \notin M$,
(xii) $a_1, a_2 \perp b_1, b_2$,
(xiii) $a_2, a_3 \perp b_2, b_3$,
(xiv) $a_3, a_4 \perp b_3, b_4$.
Then $a_1, a_4 \perp b_1, b_4$.

Let us consider X. We say that MH2 holds in X if and only if the condition (Def.4) is satisfied.

(Def.4) Given a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, M, N. Suppose that
(i) $M \perp N$,
(ii) $a_1 \in M$,
(iii) $a_3 \in M$,
(iv) $b_2 \in M$,
(v) $b_4 \in M$,
(vi) $a_2 \in N$,
(vii) $a_4 \in N$,
(viii) $b_1 \in N$,
(ix) $b_3 \in N$,
(x) $a_2 \notin M$,
(xi) $a_4 \notin M$,
(xii) $a_1, a_2 \perp b_1, b_2$,
(xiii) $a_2, a_3 \perp b_2, b_3$,
(xiv) $a_3, a_4 \perp b_3, b_4$.
Then $a_1, a_4 \perp b_1, b_4$.

Let us consider X. We say that trapezium variant of Desargues Axiom holds in X if and only if the condition (Def.5) is satisfied.

(Def.5) Given o, a, a_1, b, b_1, c, c_1. Suppose that
(i) $o \neq a$,
(ii) $o \neq a_1$,
(iii) $o \neq b$,
(iv) $o \neq b_1$,
(v) $o \neq c$,
(vi) $o \neq c_1$,
(vii) not $\mathbf{L}(b, b_1, a)$,
(viii) not $\mathbf{L}(b, b_1, c)$,
(ix) $\mathbf{L}(o, a, a_1)$,
(x) $\mathbf{L}(o, b, b_1)$,
(xi) $\mathbf{L}(o, c, c_1)$,
(xii) $a, b \parallel a_1, b_1$,
(xiii) $a, b \parallel o, c$,
(xiv) $b, c \parallel b_1, c_1$.
Then $a, c \parallel a_1, c_1$.

Let us consider X. We say that Scherungssatz holds in X if and only if the condition (Def.6) is satisfied.

(Def.6) Given a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, M, N. Suppose that

(i) M is a line,
(ii) N is a line,
(iii) $a_1 \in M$,
(iv) $a_3 \in M$,
(v) $b_1 \in M$,
(vi) $b_3 \in M$,
(vii) $a_2 \in N$,
(viii) $a_4 \in N$,
(ix) $b_2 \in N$,
(x) $b_4 \in N$,
(xi) $a_4 \notin M$,
(xii) $a_2 \notin M$,
(xiii) $b_2 \notin M$,
(xiv) $b_4 \notin M$,
(xv) $a_1 \notin N$,
(xvi) $a_3 \notin N$,
(xvii) $b_1 \notin N$,
(xviii) $b_3 \notin N$,
(xix) $a_3, a_2 \parallel b_3, b_2$,
(xx) $a_2, a_1 \parallel b_2, b_1$,
(xxi) $a_1, a_4 \parallel b_1, b_4$.

Then $a_3, a_4 \parallel b_3, b_4$.

Let us consider X. We say that Scherungssatz with orthogonal axes holds in X if and only if the condition (Def.7) is satisfied.

(Def.7) Given a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, M, N. Suppose that

(i) $M \perp N$,
(ii) $a_1 \in M$,
(iii) $a_3 \in M$,
(iv) $b_1 \in M$,
(v) $b_3 \in M$,
(vi) $a_2 \in N$,
(vii) $a_4 \in N$,
(viii) $b_2 \in N$,
(ix) $b_4 \in N$,
(x) $a_4 \notin M$,
(xi) $a_2 \notin M$,
(xii) $b_2 \notin M$,
(xiii) $b_4 \notin M$,
(xiv) $a_1 \notin N$,
(xv) $a_3 \notin N$,
(xvi) $b_1 \notin N$,

(xvii) $b_3 \notin N$,

(xviii) $a_3, a_2 \parallel b_3, b_2$,

(xix) $a_2, a_1 \parallel b_2, b_1$,

(xx) $a_1, a_4 \parallel b_1, b_4$.

Then $a_3, a_4 \parallel b_3, b_4$.

Let us consider X. We say that minor Desargues Axiom holds in X if and only if:

(Def.8) for all a, a_1, b, b_1, c, c_1 such that not $\mathbf{L}(a, a_1, b)$ and not $\mathbf{L}(a, a_1, c)$ and $a, a_1 \parallel b, b_1$ and $a, a_1 \parallel c, c_1$ and $a, b \parallel a_1, b_1$ and $a, c \parallel a_1, c_1$ holds $b, c \parallel b_1, c_1$.

One can prove the following propositions:

(1) There exist a, b, c such that $\mathbf{L}(a, b, c)$ and $a \neq b$ and $b \neq c$ and $c \neq a$.

(2) For all a, b such that $a \neq b$ there exists c such that $\mathbf{L}(a, b, c)$ and $a \neq c$ and $b \neq c$.

(3) For all A, a such that A is a line there exists K such that $a \in K$ and $A \perp K$.

(4) If A is a line and $a \in A$ and $b \in A$ and $c \in A$, then $\mathbf{L}(a, b, c)$.

(5) If A is a line and M is a line and $a \in A$ and $b \in A$ and $a \in M$ and $b \in M$, then $a = b$ or $A = M$.

(6) For all a, b, c, d, M and for every subset M' of the points of the affine reduct of X and for all elements c', d' of the points of the affine reduct of X such that $c = c'$ and $d = d'$ and $M = M'$ and $a \in M$ and $b \in M$ and $c', d' \parallel M'$ holds $c, d \parallel a, b$.

(7) If trapezium variant of Desargues Axiom holds in X, then the affine reduct of X satisfies **TDES**.

(8) If the affine reduct of X satisfies **des**, then minor Desargues Axiom holds in X.

(9) If MH1 holds in X, then Scherungssatz with orthogonal axes holds in X.

(10) If MH2 holds in X, then Scherungssatz with orthogonal axes holds in X.

(11) If AH holds in X, then trapezium variant of Desargues Axiom holds in X.

(12) If Scherungssatz with orthogonal axes holds in X and trapezium variant of Desargues Axiom holds in X, then Scherungssatz holds in X.

(13) If Pappos Axiom with orthogonal axes holds in X and Desargues Axiom holds in X, then Pappos Axiom holds in X.

(14) If MH1 holds in X and MH2 holds in X, then Pappos Axiom with orthogonal axes holds in X.

(15) If theorem on three perpendiculars holds in X, then Pappos Axiom with orthogonal axes holds in X.

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Fanoian, Pappian and Desarguesian Affine Spaces

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Summary. We introduce basic types of affine spaces such as Desarguesian, Fanoian, Pappian, and translation affine and ordered affine spaces and we prove that suitably chosen analytically defined affine structures satify the required properties.

MML Identifier: `PAPDESAF`.

The articles [6], [1], [4], [5], [2], and [3] provide the notation and terminology for this paper. Let O_1 be an ordered affine space. Then $\Lambda(O_1)$ is an affine space.

Let O_1 be an ordered affine plane. Then $\Lambda(O_1)$ is an affine plane.

We now state several propositions:

(1) There exists a real linear space V and there exist vectors u, v of V such that for all real numbers a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$.

(2) For every ordered affine space O_1 and for an arbitrary x holds x is an element of the points of O_1 if and only if x is an element of the points of $\Lambda(O_1)$ but x is a subset of the points of O_1 if and only if x is a subset of the points of $\Lambda(O_1)$.

(3) For every ordered affine space O_1 and for all elements a, b, c of the points of O_1 and for all elements a', b', c' of the points of $\Lambda(O_1)$ such that $a = a'$ and $b = b'$ and $c = c'$ holds $\mathbf{L}(a, b, c)$ if and only if $\mathbf{L}(a', b', c')$.

(4) For every real linear space V and for an arbitrary x holds x is an element of the points of OASpace V if and only if x is a vector of V.

(5) Let V be a real linear space. Then for every ordered affine space O_1 such that $O_1 =$ OASpace V for all elements a, b, c, d of the points of O_1

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and for all vectors u, v, w, y of V such that $a = u$ and $b = v$ and $c = w$ and $d = y$ holds $a, b \parallel c, d$ if and only if $u, v \parallel w, y$.

(6) For every real linear space V and for every ordered affine space O_1 such that $O_1 = \mathrm{OASpace}\, V$ there exist vectors u, v of V such that for all real numbers a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$.

Let A_1 be an affine space. We say that A_1 satisfies **PAP'** if and only if the condition (Def.1) is satisfied.

(Def.1) Let M, N be subsets of the points of A_1. Let o, a, b, c, a', b', c' be elements of the points of A_1. Suppose that

(i) M is a line,
(ii) N is a line,
(iii) $M \neq N$,
(iv) $o \in M$,
(v) $o \in N$,
(vi) $o \neq a$,
(vii) $o \neq a'$,
(viii) $o \neq b$,
(ix) $o \neq b'$,
(x) $o \neq c$,
(xi) $o \neq c'$,
(xii) $a \in M$,
(xiii) $b \in M$,
(xiv) $c \in M$,
(xv) $a' \in N$,
(xvi) $b' \in N$,
(xvii) $c' \in N$,
(xviii) $a, b' \parallel b, a'$,
(xix) $b, c' \parallel c, b'$.

Then $a, c' \parallel c, a'$.

Let A_1 be an affine space. We say that A_1 satisfies **DES'** if and only if the condition (Def.2) is satisfied.

(Def.2) Let A, P, C be subsets of the points of A_1. Let o, a, b, c, a', b', c' be elements of the points of A_1. Suppose that

(i) $o \in A$,
(ii) $o \in P$,
(iii) $o \in C$,
(iv) $o \neq a$,
(v) $o \neq b$,
(vi) $o \neq c$,
(vii) $a \in A$,
(viii) $a' \in A$,
(ix) $b \in P$,
(x) $b' \in P$,
(xi) $c \in C$,

(xii) $c' \in C$,
(xiii) A is a line,
(xiv) P is a line,
(xv) C is a line,
(xvi) $A \neq P$,
(xvii) $A \neq C$,
(xviii) $a, b \parallel a', b'$,
(xix) $a, c \parallel a', c'$.
Then $b, c \parallel b', c'$.

Let A_1 be an affine space. We say that A_1 satisfies **TDES'** if and only if the condition (Def.3) is satisfied.

(Def.3) Let K be a subset of the points of A_1. Let o, a, b, c, a', b', c' be elements of the points of A_1. Suppose that
(i) K is a line,
(ii) $o \in K$,
(iii) $c \in K$,
(iv) $c' \in K$,
(v) $a \notin K$,
(vi) $o \neq c$,
(vii) $a \neq b$,
(viii) $\mathbf{L}(o, a, a')$,
(ix) $\mathbf{L}(o, b, b')$,
(x) $a, b \parallel a', b'$,
(xi) $a, c \parallel a', c'$,
(xii) $a, b \parallel K$.
Then $b, c \parallel b', c'$.

Let A_1 be an affine space. We say that A_1 satisfies **des'** if and only if the condition (Def.4) is satisfied.

(Def.4) Let A, P, C be subsets of the points of A_1. Let a, b, c, a', b', c' be elements of the points of A_1. Suppose that
(i) $A \parallel P$,
(ii) $A \parallel C$,
(iii) $a \in A$,
(iv) $a' \in A$,
(v) $b \in P$,
(vi) $b' \in P$,
(vii) $c \in C$,
(viii) $c' \in C$,
(ix) A is a line,
(x) P is a line,
(xi) C is a line,
(xii) $A \neq P$,
(xiii) $A \neq C$,
(xiv) $a, b \parallel a', b'$,

(xv) $a, c \parallel a', c'$.
Then $b, c \parallel b', c'$.

Let A_1 be an affine space. We say that A_1 satisfies Fano Axiom if and only if:

(Def.5) for all elements a, b, c, d of the points of A_1 such that $a, b \parallel c, d$ and $a, c \parallel b, d$ and $a, d \parallel b, c$ holds $a, b \parallel a, c$.

One can prove the following propositions:

(7) For every affine plane A_1 holds A_1 satisfies **PAP** if and only if A_1 satisfies **PAP'**.

(8) For every affine plane A_1 holds A_1 satisfies **DES** if and only if A_1 satisfies **DES'**.

(9) For every affine plane A_1 holds A_1 satisfies **TDES** if and only if A_1 satisfies **TDES'**.

(10) For every affine plane A_1 holds A_1 satisfies **des** if and only if A_1 satisfies **des'**.

An affine space is Pappian if:

(Def.6) it satisfies **PAP'**.

An affine space is Desarguesian if:

(Def.7) it satisfies **DES'**.

An affine space is Moufangian if:

(Def.8) it satisfies **TDES'**.

An affine space is translation if:

(Def.9) it satisfies **des'**.

An affine space is Fanoian if:

(Def.10) it satisfies Fano Axiom.

An ordered affine space is Pappian if:

(Def.11) Λ(it) satisfies **PAP'**.

An ordered affine space is Desarguesian if:

(Def.12) Λ(it) satisfies **DES'**.

An ordered affine space is Moufangian if:

(Def.13) Λ(it) satisfies **TDES'**.

An ordered affine space is translation if:

(Def.14) Λ(it) satisfies **des'**.

Let O_1 be an ordered affine space. We say that O_1 satisfies **DES** if and only if the condition (Def.15) is satisfied.

(Def.15) Let o, a, b, c, a_1, b_1, c_1 be elements of the points of O_1. Then if $o, a \upuparrows o, a_1$ and $o, b \upuparrows o, b_1$ and $o, c \upuparrows o, c_1$ and not $\mathbf{L}(o, a, b)$ and not $\mathbf{L}(o, a, c)$ and $a, b \upuparrows a_1, b_1$ and $a, c \upuparrows a_1, c_1$, then $b, c \upuparrows b_1, c_1$.

Let O_1 be an ordered affine space. We say that O_1 satisfies $\mathbf{DES_1}$ if and only if the condition (Def.16) is satisfied.

(Def.16) Let o, a, b, c, a_1, b_1, c_1 be elements of the points of O_1. Then if $a, o \upuparrows o, a_1$ and $b, o \upuparrows o, b_1$ and $c, o \upuparrows o, c_1$ and not $\mathbf{L}(o, a, b)$ and not $\mathbf{L}(o, a, c)$ and $a, b \upuparrows b_1, a_1$ and $a, c \upuparrows c_1, a_1$, then $b, c \upuparrows c_1, b_1$.

One can prove the following propositions:

(11) For every ordered affine space O_1 such that O_1 satisfies $\mathbf{DES_1}$ holds O_1 satisfies **DES**.

(12) For every ordered affine space O_1 and for all elements o, a, b, a', b' of the points of O_1 such that not $\mathbf{L}(o, a, b)$ and $a, o \upuparrows o, a'$ and $\mathbf{L}(o, b, b')$ and $a, b \parallel a', b'$ holds $b, o \upuparrows o, b'$ and $a, b \upuparrows b', a'$.

(13) For every ordered affine space O_1 and for all elements o, a, b, a', b' of the points of O_1 such that not $\mathbf{L}(o, a, b)$ and $o, a \upuparrows o, a'$ and $\mathbf{L}(o, b, b')$ and $a, b \parallel a', b'$ holds $o, b \upuparrows o, b'$ and $a, b \upuparrows a', b'$.

(14) For every ordered affine space O_2 such that O_2 satisfies $\mathbf{DES_1}$ holds $\Lambda(O_2)$ satisfies **DES'**.

(15) Let V be a real linear space. Let o, u, v, u_1, v_1 be vectors of V. Let r be a real number. Suppose $o - u = r \cdot (u_1 - o)$ and $r \neq 0$ and $o, v \parallel o, v_1$ and $o, u \nparallel o, v$ and $u, v \parallel u_1, v_1$. Then $v_1 = u_1 + (-r)^{-1} \cdot (v - u)$ and $v_1 = o + (-r)^{-1} \cdot (v - o)$ and $v - u = (-r) \cdot (v_1 - u_1)$.

(16) For every real number r such that $r \neq 0$ holds $(-r)^{-1} = -r^{-1}$.

(17) For every real linear space V and for every ordered affine space O_1 such that $O_1 = \mathrm{OASpace}\, V$ holds O_1 satisfies $\mathbf{DES_1}$.

(18) For every real linear space V and for every ordered affine space O_1 such that $O_1 = \mathrm{OASpace}\, V$ holds O_1 satisfies $\mathbf{DES_1}$ and O_1 satisfies **DES**.

(19) For every real linear space V and for every ordered affine space O_1 such that $O_1 = \mathrm{OASpace}\, V$ holds $\Lambda(O_1)$ satisfies **PAP'**.

(20) For every real linear space V and for every ordered affine space O_1 such that $O_1 = \mathrm{OASpace}\, V$ holds $\Lambda(O_1)$ satisfies **DES'**.

(21) For every affine space A_1 such that A_1 satisfies **DES'** holds A_1 satisfies **TDES'**.

(22) For every real linear space V and for every ordered affine space O_1 such that $O_1 = \mathrm{OASpace}\, V$ holds $\Lambda(O_1)$ satisfies **TDES'**.

(23) For every real linear space V and for every ordered affine space O_1 such that $O_1 = \mathrm{OASpace}\, V$ holds $\Lambda(O_1)$ satisfies **des'**.

(24) For every ordered affine space O_1 holds $\Lambda(O_1)$ satisfies Fano Axiom.

Let O_1 be an ordered affine space. Then $\Lambda(O_1)$ is an Fanoian affine space.

Let O_1 be a Pappian ordered affine space. Then $\Lambda(O_1)$ is a Pappian Fanoian affine space.

Let O_1 be a Desarguesian ordered affine space. Then $\Lambda(O_1)$ is an Desarguesian Fanoian affine space.

Let O_1 be a Moufangian ordered affine space. Then $\Lambda(O_1)$ is an Moufangian Fanoian affine space.

Let O_1 be a translation ordered affine space. Then $\Lambda(O_1)$ is a translation Fanoian affine space.

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Elementary Variants of Affine Configurational Theorems [1]

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Summary. We present elementary versions of Pappus, Major Desargues and Minor Desargues Axioms (i.e. statements formulated entirely in the language of points and parallelism of segments). Evidently they are consequences of appropriate configurational axioms introduced in the article [2]. In particular it follows that there exists an affine plane satisfying all of them.

MML Identifier: PARDEPAP.

The terminology and notation used in this paper have been introduced in the following papers: [1], [3], [2], and [4]. In the sequel S_1 will be an affine plane. The following propositions are true:

(1) If S_1 satisfies **PAP**, then for all elements a_1, a_2, a_3, b_1, b_2, b_3 of the points of S_1 such that $a_1, a_2 \parallel a_1, a_3$ and $b_1, b_2 \parallel b_1, b_3$ and $a_1, b_2 \parallel a_2, b_1$ and $a_2, b_3 \parallel a_3, b_2$ holds $a_3, b_1 \parallel a_1, b_3$.

(2) Suppose S_1 satisfies **DES**. Let o, a, a', b, b', c, c' be elements of the points of S_1. Then if $o, a \nparallel o, b$ and $o, a \nparallel o, c$ and $o, a \parallel o, a'$ and $o, b \parallel o, b'$ and $o, c \parallel o, c'$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$, then $b, c \parallel b', c'$.

(3) Suppose S_1 satisfies **des**. Let a, a', b, b', c, c' be elements of the points of S_1. Then if $a, a' \nparallel a, b$ and $a, a' \nparallel a, c$ and $a, a' \parallel b, b'$ and $a, a' \parallel c, c'$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$, then $b, c \parallel b', c'$.

(4) If S_1 satisfies Fano Axiom, then for all elements a, b, c, d of the points of S_1 such that $a, b \nparallel a, c$ and $a, b \parallel c, d$ and $a, c \parallel b, d$ holds $a, d \nparallel b, c$.

(5) There exists S_1 such that for all elements o, a, a', b, b', c, c' of the points of S_1 such that $o, a \nparallel o, b$ and $o, a \nparallel o, c$ and $o, a \parallel o, a'$ and $o, b \parallel o, b'$ and $o, c \parallel o, c'$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ holds $b, c \parallel b', c'$ and for

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all elements a, a', b, b', c, c' of the points of S_1 such that $a, a' \nparallel a, b$ and $a, a' \nparallel a, c$ and $a, a' \parallel b, b'$ and $a, a' \parallel c, c'$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ holds $b, c \parallel b', c'$ and for all elements a_1, a_2, a_3, b_1, b_2, b_3 of the points of S_1 such that $a_1, a_2 \parallel a_1, a_3$ and $b_1, b_2 \parallel b_1, b_3$ and $a_1, b_2 \parallel a_2, b_1$ and $a_2, b_3 \parallel a_3, b_2$ holds $a_3, b_1 \parallel a_1, b_3$ and for all elements a, b, c, d of the points of S_1 such that $a, b \nparallel a, c$ and $a, b \parallel c, d$ and $a, c \parallel b, d$ holds $a, d \nparallel b, c$.

(6) For every elements o, a of the points of S_1 there exists an element p of the points of S_1 such that for all elements b, c of the points of S_1 holds $o, a \parallel o, p$ and there exists an element d of the points of S_1 such that if $o, p \parallel o, b$, then $o, c \parallel o, d$ and $p, c \parallel b, d$.

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Semi-Affine Space [1]

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Summary. A brief survey on semi-affine geometry, which results from the classical Pappian and Desarguesian affine (dimension free) geometry by weakening the so called trapezium axiom. With the help of the relation of parallelogram in every semi-affine space we define the operation of "addition" of "vectors". Next we investigate in greater details the relation of (affine) trapezium in such spaces.

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The papers [3], [2], and [1] provide the notation and terminology for this paper. An affine structure is called a semi affine space if it satisfies the conditions (Def.1).

(Def.1) (i) For all elements a, b of the points of it holds $a, b \parallel b, a$,

(ii) for all elements a, b, c of the points of it holds $a, b \parallel c, c$,

(iii) for all elements a, b, p, q, r, s of the points of it such that $a \neq b$ and $a, b \parallel p, q$ and $a, b \parallel r, s$ holds $p, q \parallel r, s$,

(iv) for all elements a, b, c of the points of it such that $a, b \parallel a, c$ holds $b, a \parallel b, c$,

(v) there exist elements a, b, c of the points of it such that $a, b \nparallel a, c$,

(vi) for every elements a, b, p of the points of it there exists an element q of the points of it such that $a, b \parallel p, q$ and $a, p \parallel b, q$,

(vii) for every elements o, a of the points of it there exists an element p of the points of it such that for all elements b, c of the points of it holds $o, a \parallel o, p$ and there exists an element d of the points of it such that if $o, p \parallel o, b$, then $o, c \parallel o, d$ and $p, c \parallel b, d$,

(viii) for all elements o, a, a', b, b', c, c' of the points of it such that $o, a \nparallel o, b$ and $o, a \nparallel o, c$ and $o, a \parallel o, a'$ and $o, b \parallel o, b'$ and $o, c \parallel o, c'$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ holds $b, c \parallel b', c'$,

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(ix) for all elements a, a', b, b', c, c' of the points of it such that $a, a' \nparallel a, b$ and $a, a' \nparallel a, c$ and $a, a' \parallel b, b'$ and $a, a' \parallel c, c'$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$ holds $b, c \parallel b', c'$,

(x) for all elements a_1, a_2, a_3, b_1, b_2, b_3 of the points of it such that $a_1, a_2 \parallel a_1, a_3$ and $b_1, b_2 \parallel b_1, b_3$ and $a_1, b_2 \parallel a_2, b_1$ and $a_2, b_3 \parallel a_3, b_2$ holds $a_3, b_1 \parallel a_1, b_3$,

(xi) for all elements a, b, c, d of the points of it such that $a, b \nparallel a, c$ and $a, b \parallel c, d$ and $a, c \parallel b, d$ holds $a, d \nparallel b, c$.

We adopt the following convention: S_1 will be a semi affine space and a, a', a_1, a_2, a_3, a_4, b, b', b_1, b_2, b_3, c, c', d, d', d_1, d_2, o, p, p_1, p_2, q, r, r_1, r_2, s, x, y, z will be elements of the points of S_1. The following propositions are true:

(1) $a, b \parallel b, a$.

(2) $a, b \parallel c, c$.

(3) If $a \neq b$ and $a, b \parallel p, q$ and $a, b \parallel r, s$, then $p, q \parallel r, s$.

(4) If $a, b \parallel a, c$, then $b, a \parallel b, c$.

(5) There exist a, b, c such that $a, b \nparallel a, c$.

(6) There exists q such that $a, b \parallel p, q$ and $a, p \parallel b, q$.

(7) For every o, a there exists p such that for all b, c holds $o, a \parallel o, p$ and there exists d such that if $o, p \parallel o, b$, then $o, c \parallel o, d$ and $p, c \parallel b, d$.

(8) If $o, a \nparallel o, b$ and $o, a \nparallel o, c$ and $o, a \parallel o, a'$ and $o, b \parallel o, b'$ and $o, c \parallel o, c'$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$, then $b, c \parallel b', c'$.

(9) If $a, a' \nparallel a, b$ and $a, a' \nparallel a, c$ and $a, a' \parallel b, b'$ and $a, a' \parallel c, c'$ and $a, b \parallel a', b'$ and $a, c \parallel a', c'$, then $b, c \parallel b', c'$.

(10) If $a_1, a_2 \parallel a_1, a_3$ and $b_1, b_2 \parallel b_1, b_3$ and $a_1, b_2 \parallel a_2, b_1$ and $a_2, b_3 \parallel a_3, b_2$, then $a_3, b_1 \parallel a_1, b_3$.

(11) If $a, b \nparallel a, c$ and $a, b \parallel c, d$ and $a, c \parallel b, d$, then $a, d \nparallel b, c$.

(12) $a, b \parallel a, b$.

(13) If $a, b \parallel c, d$, then $c, d \parallel a, b$.

(14) $a, a \parallel b, c$.

(15) If $a, b \parallel c, d$, then $b, a \parallel c, d$.

(16) If $a, b \parallel c, d$, then $a, b \parallel d, c$.

(17) If $a, b \parallel c, d$, then $b, a \parallel c, d$ and $a, b \parallel d, c$ and $b, a \parallel d, c$ and $c, d \parallel a, b$ and $d, c \parallel a, b$ and $c, d \parallel b, a$ and $d, c \parallel b, a$.

(18) Suppose $a, b \parallel a, c$. Then $a, c \parallel a, b$ and $b, a \parallel a, c$ and $a, b \parallel c, a$ and $a, c \parallel b, a$ and $b, a \parallel c, a$ and $c, a \parallel a, b$ and $c, a \parallel b, a$ and $b, a \parallel b, c$ and $a, b \parallel b, c$ and $b, a \parallel c, b$ and $b, c \parallel b, a$ and $a, b \parallel c, b$ and $c, b \parallel b, a$ and $b, c \parallel a, b$ and $c, b \parallel a, b$ and $c, a \parallel c, b$ and $a, c \parallel c, b$ and $c, a \parallel b, c$ and $a, c \parallel b, c$ and $c, b \parallel c, a$ and $b, c \parallel c, a$ and $c, b \parallel a, c$ and $b, c \parallel a, c$.

(19) If $a, b \parallel p, q$ and $a, b \parallel r, s$, then $a = b$ or $p, q \parallel r, s$.

(20) If $a \neq b$ and $p, q \parallel a, b$ and $a, b \parallel r, s$, then $p, q \parallel r, s$.

(21) If $a, b \nparallel a, d$, then $a \neq b$ and $b \neq d$ and $d \neq a$.

(22) If $a, b \nparallel p, q$, then $a \neq b$ and $p \neq q$.

(23) If $a, b \parallel a, x$ and $b, c \parallel b, x$ and $c, a \parallel c, x$, then $a, b \parallel a, c$.

(24) If $a, b \nparallel a, c$, then $a, b \nparallel a, x$ or $b, c \nparallel b, x$ or $c, a \nparallel c, x$.

(25) If $a, b \nparallel a, c$ and $p \neq q$, then $p, q \nparallel p, a$ or $p, q \nparallel p, b$ or $p, q \nparallel p, c$.

(26) If $p \neq q$, then there exists r such that $p, q \nparallel p, r$.

(27) Suppose $a, b \nparallel c, d$. Then $a, b \nparallel d, c$ and $b, a \nparallel c, d$ and $b, a \nparallel d, c$ and $c, d \nparallel a, b$ and $c, d \nparallel b, a$ and $d, c \nparallel a, b$ and $d, c \nparallel b, a$.

(28) Suppose $a, b \nparallel a, c$. Then $a, b \nparallel c, a$ and $b, a \nparallel a, c$ and $b, a \nparallel c, a$ and $a, c \nparallel a, b$ and $a, c \nparallel b, a$ and $c, a \nparallel a, b$ and $c, a \nparallel b, a$ and $b, a \nparallel b, c$ and $b, a \nparallel c, b$ and $a, b \nparallel b, c$ and $a, b \nparallel c, b$ and $b, c \nparallel b, a$ and $b, c \nparallel a, b$ and $c, b \nparallel a, b$ and $c, b \nparallel b, a$ and $c, b \nparallel c, a$ and $c, b \nparallel a, c$ and $b, c \nparallel c, a$ and $b, c \nparallel a, c$ and $c, a \nparallel c, b$ and $c, a \nparallel b, c$ and $a, c \nparallel b, c$ and $a, c \nparallel c, b$.

(29) If $a, b \nparallel c, d$ and $a, b \parallel p, q$ and $c, d \parallel r, s$ and $p \neq q$ and $r \neq s$, then $p, q \nparallel r, s$.

(30) If $a, b \nparallel a, c$ and $a, b \parallel p, q$ and $a, c \parallel p, r$ and $b, c \parallel q, r$ and $p \neq q$, then $p, q \nparallel p, r$.

(31) If $a, b \nparallel a, c$ and $a, c \parallel p, r$ and $b, c \parallel p, r$, then $p = r$.

We now state four propositions:

(32) If $p, q \nparallel p, r_1$ and $p, r_1 \parallel p, r_2$ and $q, r_1 \parallel q, r_2$, then $r_1 = r_2$.

(33) If $a, b \nparallel a, c$ and $a, b \parallel p, q$ and $a, c \parallel p, r_1$ and $a, c \parallel p, r_2$ and $b, c \parallel q, r_1$ and $b, c \parallel q, r_2$, then $r_1 = r_2$.

(34) If $a = b$ or $c = d$ or $a = c$ and $b = d$ or $a = d$ and $b = c$, then $a, b \parallel c, d$.

(35) If $a = b$ or $a = c$ or $b = c$, then $a, b \parallel a, c$.

Let us consider S_1, a, b, c. We say that a, b and c are collinear if and only if:

(Def.2) $a, b \parallel a, c$.

We now state a number of propositions:

(37)[2] If a_1, a_2 and a_3 are collinear, then a_1, a_3 and a_2 are collinear and a_2, a_1 and a_3 are collinear and a_2, a_3 and a_1 are collinear and a_3, a_1 and a_2 are collinear and a_3, a_2 and a_1 are collinear.

(38) If a_1, a_2 and a_3 are not collinear, then a_1, a_3 and a_2 are not collinear and a_2, a_1 and a_3 are not collinear and a_2, a_3 and a_1 are not collinear and a_3, a_1 and a_2 are not collinear and a_3, a_2 and a_1 are not collinear.

(39) If a, b and c are not collinear and $a, b \parallel p, q$ and $a, c \parallel p, r$ and $p \neq q$ and $p \neq r$, then p, q and r are not collinear.

(40) If $a = b$ or $b = c$ or $c = a$, then a, b and c are collinear.

(41) If $p \neq q$, then there exists r such that p, q and r are not collinear.

(42) If a, b and c are collinear and a, b and d are collinear, then $a, b \parallel c, d$.

(43) If a, b and c are not collinear and $a, b \parallel c, d$, then a, b and d are not collinear.

[2]The proposition (36) was either repeated or obvious.

(44) If a, b and c are not collinear and $a, b \parallel c, d$ and $c \neq d$ and c, d and x are collinear, then a, b and x are not collinear.

(45) If o, a and b are not collinear and o, a and x are collinear and o, b and x are collinear, then $o = x$.

(46) If $o \neq a$ and $o \neq b$ and o, a and b are collinear and o, a and a' are collinear and o, b and b' are collinear, then $a, b \parallel a', b'$.

(48)[3] If $a, b \nparallel c, d$ and a, b and p_1 are collinear and a, b and p_2 are collinear and c, d and p_1 are collinear and c, d and p_2 are collinear, then $p_1 = p_2$.

(49) If $a \neq b$ and a, b and c are collinear and $a, b \parallel c, d$, then $a, c \parallel b, d$.

(50) If $a \neq b$ and a, b and c are collinear and $a, b \parallel c, d$, then $c, b \parallel c, d$.

(51) If o, a and c are not collinear and o, a and b are collinear and o, c and d_1 are collinear and o, c and d_2 are collinear and $a, c \parallel b, d_1$ and $a, c \parallel b, d_1$ and $a, c \parallel b, d_2$, then $d_1 = d_2$.

(52) If $a \neq b$ and a, b and c are collinear and a, b and d are collinear, then a, c and d are collinear.

Let us consider S_1, a, b, c, d. We say that a, b, c, d form a parallelogram if and only if:

(Def.3) a, b and c are not collinear and $a, b \parallel c, d$ and $a, c \parallel b, d$.

We now state a number of propositions:

(54)[4] If a, b, c, d form a parallelogram, then $a \neq b$ and $a \neq c$ and $c \neq b$ and $a \neq d$ and $b \neq d$ and $c \neq d$.

(55) If a, b, c, d form a parallelogram, then a, b and c are not collinear and b, a and d are not collinear and c, d and a are not collinear and d, c and b are not collinear.

(56) Suppose a_1, a_2, a_3, a_4 form a parallelogram. Then a_1, a_2 and a_3 are not collinear and a_1, a_3 and a_2 are not collinear and a_1, a_2 and a_4 are not collinear and a_1, a_4 and a_2 are not collinear and a_1, a_3 and a_4 are not collinear and a_1, a_4 and a_3 are not collinear and a_2, a_1 and a_3 are not collinear and a_2, a_3 and a_1 are not collinear and a_2, a_1 and a_4 are not collinear and a_2, a_4 and a_1 are not collinear and a_2, a_3 and a_4 are not collinear and a_2, a_4 and a_3 are not collinear and a_3, a_1 and a_2 are not collinear and a_3, a_2 and a_1 are not collinear and a_3, a_1 and a_4 are not collinear and a_3, a_4 and a_1 are not collinear and a_3, a_2 and a_4 are not collinear and a_3, a_4 and a_2 are not collinear and a_4, a_1 and a_2 are not collinear and a_4, a_2 and a_1 are not collinear and a_4, a_1 and a_3 are not collinear and a_4, a_3 and a_1 are not collinear and a_4, a_2 and a_3 are not collinear and a_4, a_3 and a_2 are not collinear.

(57) If a, b, c, d form a parallelogram, then a, b and x are not collinear or c, d and x are not collinear.

(58) If a, b, c, d form a parallelogram, then a, c, b, d form a parallelogram.

[3]The proposition (47) was either repeated or obvious.
[4]The proposition (53) was either repeated or obvious.

(59) If a, b, c, d form a parallelogram, then c, d, a, b form a parallelogram.

(60) If a, b, c, d form a parallelogram, then b, a, d, c form a parallelogram.

(61) If a, b, c, d form a parallelogram, then a, c, b, d form a parallelogram and c, d, a, b form a parallelogram and b, a, d, c form a parallelogram and c, a, d, b form a parallelogram and d, b, c, a form a parallelogram and b, d, a, c form a parallelogram.

(62) If a, b and c are not collinear, then there exists d such that a, b, c, d form a parallelogram.

(63) If a, b, c, d_1 form a parallelogram and a, b, c, d_2 form a parallelogram, then $d_1 = d_2$.

(64) If a, b, c, d form a parallelogram, then $a, d \nparallel b, c$.

(65) If a, b, c, d form a parallelogram, then a, b, d, c do not form a parallelogram.

(66) If $a \neq b$, then there exists c such that a, b and c are collinear and $c \neq a$ and $c \neq b$.

(67) If a, a', b, b' form a parallelogram and a, a', c, c' form a parallelogram, then $b, c \parallel b', c'$.

(68) If b, b' and c are not collinear and a, a', b, b' form a parallelogram and a, a', c, c' form a parallelogram, then b, b', c, c' form a parallelogram.

(69) If a, b and c are collinear and $b \neq c$ and a, a', b, b' form a parallelogram and a, a', c, c' form a parallelogram, then b, b', c, c' form a parallelogram.

(70) If a, a', b, b' form a parallelogram and a, a', c, c' form a parallelogram and b, b', d, d' form a parallelogram, then $c, d \parallel c', d'$.

(71) If $a \neq d$, then there exist b, c such that a, b, c, d form a parallelogram.

Let us consider S_1, a, b, r, s. We say that a, b are congruent to r, s if and only if:

(Def.4) $a = b$ and $r = s$ or there exist p, q such that p, q, a, b form a parallelogram and p, q, r, s form a parallelogram.

Next we state a number of propositions:

(73)[5] If a, a are congruent to b, c, then $b = c$.

(74) If a, b are congruent to c, c, then $a = b$.

(75) If a, b are congruent to b, a, then $a = b$.

(76) If a, b are congruent to c, d, then $a, b \parallel c, d$.

(77) If a, b are congruent to c, d, then $a, c \parallel b, d$.

(78) If a, b are congruent to c, d and a, b and c are not collinear, then a, b, c, d form a parallelogram.

(79) If a, b, c, d form a parallelogram, then a, b are congruent to c, d.

(80) If a, b are congruent to c, d and a, b and c are collinear and r, s, a, b form a parallelogram, then r, s, c, d form a parallelogram.

[5]The proposition (72) was either repeated or obvious.

(81) If a, b are congruent to c, x and a, b are congruent to c, y, then $x = y$.

(82) There exists d such that a, b are congruent to c, d.

(83) a, a are congruent to b, b.

(84) a, b are congruent to a, b.

(85) If r, s are congruent to a, b and r, s are congruent to c, d, then a, b are congruent to c, d.

(86) If a, b are congruent to c, d, then c, d are congruent to a, b.

(87) If a, b are congruent to c, d, then b, a are congruent to d, c.

(88) If a, b are congruent to c, d, then a, c are congruent to b, d.

(89) If a, b are congruent to c, d, then c, d are congruent to a, b and b, a are congruent to d, c and a, c are congruent to b, d and d, c are congruent to b, a and b, d are congruent to a, c and c, a are congruent to d, b and d, b are congruent to c, a.

(90) If a, b are congruent to p, q and b, c are congruent to q, s, then a, c are congruent to p, s.

(91) If b, a are congruent to p, q and c, a are congruent to p, r, then b, c are congruent to r, q.

(92) If a, o are congruent to o, p and b, o are congruent to o, q, then a, b are congruent to q, p.

(93) If b, a are congruent to p, q and c, a are congruent to p, r, then $b, c \parallel q, r$.

(94) If a, o are congruent to o, p and b, o are congruent to o, q, then $a, b \parallel p, q$.

Let us consider S_1, a, b, o. The functor $\mathrm{sum}_o(a, b)$ yielding an element of the points of S_1 is defined as follows:

(Def.5) o, a are congruent to b, $\mathrm{sum}_o(a, b)$.

Next we state the proposition

(95) $\mathrm{sum}_o(a, b) = c$ if and only if o, a are congruent to b, c.

Let us consider S_1, a, o. The functor $\mathrm{opposite}_o(a)$ yields an element of the points of S_1 and is defined as follows:

(Def.6) $\mathrm{sum}_o(a, \mathrm{opposite}_o(a)) = o$.

We now state the proposition

(96) $\mathrm{opposite}_o(a) = b$ if and only if $\mathrm{sum}_o(a, b) = o$.

Let us consider S_1, a, b, o. The functor $\mathrm{diff}_o(a, b)$ yielding an element of the points of S_1 is defined as follows:

(Def.7) $\mathrm{diff}_o(a, b) = \mathrm{sum}_o(a, \mathrm{opposite}_o(b))$.

Next we state a number of propositions:

(97) $\mathrm{diff}_o(a, b) = \mathrm{sum}_o(a, \mathrm{opposite}_o(b))$.

(98) o, a are congruent to b, $\mathrm{sum}_o(a, b)$.

(99) $\mathrm{sum}_o(a, o) = a$.

(100) There exists x such that $\mathrm{sum}_o(a, x) = o$.

(101) $\mathrm{sum}_o(\mathrm{sum}_o(a, b), c) = \mathrm{sum}_o(a, \mathrm{sum}_o(b, c))$.

(102) $\mathrm{sum}_o(a,b) = \mathrm{sum}_o(b,a)$.

(103) If $\mathrm{sum}_o(a,a) = o$, then $a = o$.

(104) If $\mathrm{sum}_o(a,x) = \mathrm{sum}_o(a,y)$, then $x = y$.

(105) $\mathrm{sum}_o(a, \mathrm{opposite}_o(a)) = o$.

(106) a, o are congruent to o, $\mathrm{opposite}_o(a)$.

(107) If $\mathrm{opposite}_o(a) = \mathrm{opposite}_o(b)$, then $a = b$.

(108) $a, b \parallel \mathrm{opposite}_o(a), \mathrm{opposite}_o(b)$.

(109) $\mathrm{opposite}_o(o) = o$.

(110) $p, q \parallel \mathrm{sum}_o(p,r), \mathrm{sum}_o(q,r)$.

(111) If $p, q \parallel r, s$, then $p, q \parallel \mathrm{sum}_o(p,r), \mathrm{sum}_o(q,s)$.

(113)[6] $\mathrm{diff}_o(a,b) = o$ if and only if $a = b$.

(114) $o, \mathrm{diff}_o(b,a) \parallel a, b$.

(115) o, $\mathrm{diff}_o(b,a)$ and $\mathrm{diff}_o(d,c)$ are collinear if and only if $a, b \parallel c, d$.

Let us consider S_1, a, b, c, d, o. We say that a, b, c, d form a trapezium with vertex o if and only if:

(Def.8) o, a and c are not collinear and o, a and b are collinear and o, c and d are collinear and $a, c \parallel b, d$.

Let us consider S_1, o, p. We say that there are trapeziums through p with vertex o if and only if:

(Def.9) for every b, c there exists d such that if o, p and b are collinear, then o, c and d are collinear and $p, c \parallel b, d$.

One can prove the following propositions:

(118)[7] If a, b, c, d form a trapezium with vertex o, then $o \neq a$ and $a \neq c$ and $c \neq o$.

(119) If a, b, c, x form a trapezium with vertex o and a, b, c, y form a trapezium with vertex o, then $x = y$.

(120) If o, a and b are not collinear, then a, o, b, o form a trapezium with vertex o.

(121) If a, b, c, d form a trapezium with vertex o, then c, d, a, b form a trapezium with vertex o.

(122) If $o \neq b$ and a, b, c, d form a trapezium with vertex o, then $o \neq d$.

(123) If $o \neq b$ and a, b, c, d form a trapezium with vertex o, then o, b and d are not collinear.

(124) If $o \neq b$ and a, b, c, d form a trapezium with vertex o, then b, a, d, c form a trapezium with vertex o.

(125) If $o = b$ or $o = d$ but a, b, c, d form a trapezium with vertex o, then $o = b$ and $o = d$.

[6] The proposition (112) was either repeated or obvious.

[7] The propositions (116)–(117) were either repeated or obvious.

(126) If a, p, b, q form a trapezium with vertex o and a, p, c, r form a trapezium with vertex o, then $b, c \parallel q, r$.

(127) If a, p, b, q form a trapezium with vertex o and a, p, c, r form a trapezium with vertex o and o, b and c are not collinear, then b, q, c, r form a trapezium with vertex o.

(128) If a, p, b, q form a trapezium with vertex o and a, p, c, r form a trapezium with vertex o and b, q, d, s form a trapezium with vertex o, then $c, d \parallel r, s$.

(129) For every o, a there exists p such that o, a and p are collinear and there are trapeziums through p with vertex o.

(130) There exist x, y, z such that $x \neq y$ and $y \neq z$ and $z \neq x$.

(131) If there are trapeziums through p with vertex o, then $o \neq p$.

(132) If there are trapeziums through p with vertex o, then there exists q such that o, p and q are not collinear and there are trapeziums through q with vertex o.

(133) If o, p and c are not collinear and o, p and b are collinear and there are trapeziums through p with vertex o, then there exists d such that p, b, c, d form a trapezium with vertex o.

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Planes in Affine Spaces [1]

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Summary. We introduce the notion of plane in affine space and investigate fundamental properties of them. Further we introduce the relation of parallelism defined for arbitrary subsets. In particular we are concerned with parallelisms which hold between lines and planes and between planes. We also define a function which assigns to every line and every point the unique line passing through the point and parallel to the given line. With the help of the introduced notions we prove that every at least 3-dimensional affine space is Desarguesian and translation.

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The articles [5], [1], [2], [3], and [4] provide the notation and terminology for this paper. We follow a convention: A_1 will be an affine space, a, b, c, d, a', b', c', p, q will be elements of the points of A_1, and A, C, K, M, N, P, Q, X, Y, Z will be subsets of the points of A_1. Let us consider A_1, X, Y. Then $X \cap Y$ is a subset of the points of A_1.

The following propositions are true:

(1) If $\mathbf{L}(p, a, a')$ or $\mathbf{L}(p, a', a)$ but $p \neq a$, then there exists b' such that $\mathbf{L}(p, b, b')$ and $a, b \parallel a', b'$.

(2) If $a, b \parallel A$ or $b, a \parallel A$ but $a \in A$, then $b \in A$.

(3) If $a, b \parallel A$ or $b, a \parallel A$ but $A \parallel K$ or $K \parallel A$, then $a, b \parallel K$ and $b, a \parallel K$.

(4) If $a, b \parallel A$ or $b, a \parallel A$ but $a, b \parallel c, d$ or $c, d \parallel a, b$ and $a \neq b$, then $c, d \parallel A$ and $d, c \parallel A$.

(5) If $a, b \parallel M$ or $b, a \parallel M$ but $a, b \parallel N$ or $b, a \parallel N$ and $a \neq b$, then $M \parallel N$ and $N \parallel M$.

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(6) If $a, b \parallel M$ or $b, a \parallel M$ but $c, d \parallel M$ or $d, c \parallel M$, then $a, b \parallel c, d$ and $a, b \parallel d, c$.

(7) If $A \parallel C$ or $C \parallel A$ but $a \neq b$ but $a, b \parallel c, d$ or $c, d \parallel a, b$ and $a \in A$ and $b \in A$ and $c \in C$, then $d \in C$.

(8) Suppose that
(i) $q \in M$,
(ii) $q \in N$,
(iii) $a \in M$,
(iv) $a' \in M$,
(v) $b \in N$,
(vi) $b' \in N$,
(vii) $q \neq a$,
(viii) $q \neq b$,
(ix) $M \neq N$,
(x) $a, b \parallel a', b'$ or $b, a \parallel b', a'$,
(xi) M is a line,
(xii) N is a line,
(xiii) $q = a'$.
Then $q = b'$.

(9) Suppose that
(i) $q \in M$,
(ii) $q \in N$,
(iii) $a \in M$,
(iv) $a' \in M$,
(v) $b \in N$,
(vi) $b' \in N$,
(vii) $q \neq a$,
(viii) $q \neq b$,
(ix) $M \neq N$,
(x) $a, b \parallel a', b'$ or $b, a \parallel b', a'$,
(xi) M is a line,
(xii) N is a line,
(xiii) $a = a'$.
Then $b = b'$.

(10) If $M \parallel N$ or $N \parallel M$ but $a \in M$ and $a' \in M$ and $b \in N$ and $b' \in N$ and $M \neq N$ but $a, b \parallel a', b'$ or $b, a \parallel b', a'$ and $a = a'$, then $b = b'$.

(11) There exists A such that $a \in A$ and $b \in A$ and A is a line.

(12) If A is a line, then there exists q such that $q \notin A$.

Let us consider A_1, K, P. The functor $\mathrm{Plane}(K, P)$ yielding a subset of the points of A_1 is defined by:

(Def.1) $\mathrm{Plane}(K, P) = \{a : \bigvee_b [a, b \parallel K \wedge b \in P]\}$.

Let us consider A_1, X. We say that X is a plane if and only if:

(Def.2) there exist K, P such that K is a line and P is a line and $K \nparallel P$ and $X = \text{Plane}(K, P)$.

We now state a number of propositions:

(13) If K is not a line, then $\text{Plane}(K, P) = \emptyset$.

(14) If K is a line, then $P \subseteq \text{Plane}(K, P)$.

(15) If $K \parallel P$, then $\text{Plane}(K, P) = P$.

(16) If $K \parallel M$, then $\text{Plane}(K, P) = \text{Plane}(M, P)$.

(17) Suppose that
- (i) $p \in M$,
- (ii) $a \in M$,
- (iii) $b \in M$,
- (iv) $p \in N$,
- (v) $a' \in N$,
- (vi) $b' \in N$,
- (vii) $p \notin P$,
- (viii) $p \notin Q$,
- (ix) $M \neq N$,
- (x) $a \in P$,
- (xi) $a' \in P$,
- (xii) $b \in Q$,
- (xiii) $b' \in Q$,
- (xiv) M is a line,
- (xv) N is a line,
- (xvi) P is a line,
- (xvii) Q is a line.

Then $P \parallel Q$ or there exists q such that $q \in P$ and $q \in Q$.

(18) Suppose $a \in M$ and $b \in M$ and $a' \in N$ and $b' \in N$ and $a \in P$ and $a' \in P$ and $b \in Q$ and $b' \in Q$ and $M \neq N$ and $M \parallel N$ and P is a line and Q is a line. Then $P \parallel Q$ or there exists q such that $q \in P$ and $q \in Q$.

(19) If X is a plane and $a \in X$ and $b \in X$ and $a \neq b$, then $\text{Line}(a, b) \subseteq X$.

(20) If K is a line and P is a line and Q is a line and $K \nparallel P$ and $K \nparallel Q$ and $Q \subseteq \text{Plane}(K, P)$, then $\text{Plane}(K, Q) = \text{Plane}(K, P)$.

(21) If K is a line and P is a line and Q is a line and $K \nparallel P$ and $Q \subseteq \text{Plane}(K, P)$, then $P \parallel Q$ or there exists q such that $q \in P$ and $q \in Q$.

(22) If X is a plane and M is a line and N is a line and $M \subseteq X$ and $N \subseteq X$, then $M \parallel N$ or there exists q such that $q \in M$ and $q \in N$.

(23) If X is a plane and $a \in X$ and $M \subseteq X$ and $a \in N$ but $M \parallel N$ or $N \parallel M$, then $N \subseteq X$.

(24) If X is a plane and Y is a plane and $a \in X$ and $b \in X$ and $a \in Y$ and $b \in Y$ and $X \neq Y$ and $a \neq b$, then $X \cap Y$ is a line.

(25) If X is a plane and Y is a plane and $a \in X$ and $b \in X$ and $c \in X$ and $a \in Y$ and $b \in Y$ and $c \in Y$ and not $\mathbf{L}(a, b, c)$, then $X = Y$.

(26) If X is a plane and Y is a plane and M is a line and N is a line and $M \subseteq X$ and $N \subseteq X$ and $M \subseteq Y$ and $N \subseteq Y$ and $M \neq N$, then $X = Y$.

Let us consider A_1, a, K. Let us assume that K is a line. The functor $a \cdot K$ yields a subset of the points of A_1 and is defined by:

(Def.3) $a \in a \cdot K$ and $K \parallel a \cdot K$.

We now state several propositions:

(27) If A is a line, then $a \cdot A$ is a line.

(28) If X is a plane and M is a line and $a \in X$ and $M \subseteq X$, then $a \cdot M \subseteq X$.

(29) If X is a plane and $a \in X$ and $b \in X$ and $c \in X$ and $a, b \parallel c, d$ and $a \neq b$, then $d \in X$.

(30) If A is a line, then $a \in A$ if and only if $a \cdot A = A$.

(31) If A is a line, then $a \cdot A = a \cdot (q \cdot A)$.

(32) If $K \parallel M$, then $a \cdot K = a \cdot M$.

Let us consider A_1, X, Y. The predicate $X || Y$ is defined by:

(Def.4) for all a, A such that $a \in Y$ and A is a line and $A \subseteq X$ holds $a \cdot A \subseteq Y$.

Next we state a number of propositions:

(33) If $X \subseteq Y$ but X is a line and Y is a line or X is a plane and Y is a plane, then $X = Y$.

(34) If X is a plane, then there exist a, b, c such that $a \in X$ and $b \in X$ and $c \in X$ and not $\mathbf{L}(a, b, c)$.

(35) If M is a line and X is a plane and $M \subseteq X$, then there exists q such that $q \in X$ and $q \notin M$.

(36) For all a, A such that A is a line there exists X such that $a \in X$ and $A \subseteq X$ and X is a plane.

(37) There exists X such that $a \in X$ and $b \in X$ and $c \in X$ and X is a plane.

(38) If $q \in M$ and $q \in N$ and M is a line and N is a line, then there exists X such that $M \subseteq X$ and $N \subseteq X$ and X is a plane.

(39) If $M \parallel N$, then there exists X such that $M \subseteq X$ and $N \subseteq X$ and X is a plane.

(40) If M is a line and N is a line, then $M \parallel N$ if and only if $M || N$.

(41) If M is a line and X is a plane, then $M || X$ if and only if there exists N such that $N \subseteq X$ but $M \parallel N$ or $N \parallel M$.

(42) If M is a line and X is a plane and $M \subseteq X$, then $M || X$.

(43) If A is a line and X is a plane and $a \in A$ and $a \in X$ and $A || X$, then $A \subseteq X$.

Let us consider A_1, K, M, N. We say that K, M, N are coplanar if and only if:

(Def.5) there exists X such that $K \subseteq X$ and $M \subseteq X$ and $N \subseteq X$ and X is a plane.

The following propositions are true:

(44) If K, M, N are coplanar, then K, N, M are coplanar and M, K, N are coplanar and M, N, K are coplanar and N, K, M are coplanar and N, M, K are coplanar.

(45) If A is a line and K is a line and M is a line and N is a line and M, N, K are coplanar and M, N, A are coplanar and $M \neq N$, then M, K, A are coplanar.

(46) If K is a line and M is a line and X is a plane and $K \subseteq X$ and $M \subseteq X$ and $K \neq M$, then K, M, A are coplanar if and only if $A \subseteq X$.

(47) If $q \in K$ and $q \in M$ and K is a line and M is a line, then K, M, M are coplanar and M, K, M are coplanar and M, M, K are coplanar.

(48) If A_1 is not an affine plane and X is a plane, then there exists q such that $q \notin X$.

(49) Suppose that
(i) A_1 is not an affine plane,
(ii) $q \in A$,
(iii) $q \in P$,
(iv) $q \in C$,
(v) $q \neq a$,
(vi) $q \neq b$,
(vii) $q \neq c$,
(viii) $a \in A$,
(ix) $a' \in A$,
(x) $b \in P$,
(xi) $b' \in P$,
(xii) $c \in C$,
(xiii) $c' \in C$,
(xiv) A is a line,
(xv) P is a line,
(xvi) C is a line,
(xvii) $A \neq P$,
(xviii) $A \neq C$,
(xix) $a, b \parallel a', b'$,
(xx) $a, c \parallel a', c'$.

Then $b, c \parallel b', c'$.

(50) If A_1 is not an affine plane, then A_1 is Desarguesian.

(51) Suppose that
(i) A_1 is not an affine plane,
(ii) $A \parallel P$,
(iii) $A \parallel C$,
(iv) $a \in A$,
(v) $a' \in A$,
(vi) $b \in P$,
(vii) $b' \in P$,
(viii) $c \in C$,

(ix) $c' \in C$,
(x) A is a line,
(xi) P is a line,
(xii) C is a line,
(xiii) $A \neq P$,
(xiv) $A \neq C$,
(xv) $a, b \parallel a', b'$,
(xvi) $a, c \parallel a', c'$.
Then $b, c \parallel b', c'$.

(52) If A_1 is not an affine plane, then A_1 is translation.

(53) If A_1 is an affine plane and not $\mathbf{L}(a, b, c)$, then there exists c' such that $a, c \parallel a', c'$ and $b, c \parallel b', c'$.

(54) If not $\mathbf{L}(a, b, c)$ and $a' \neq b'$ and $a, b \parallel a', b'$, then there exists c' such that $a, c \parallel a', c'$ and $b, c \parallel b', c'$.

(55) Suppose X is a plane and Y is a plane. Then $X||Y$ if and only if there exist A, P, M, N such that $A \nparallel P$ and $A \subseteq X$ and $P \subseteq X$ and $M \subseteq Y$ and $N \subseteq Y$ but $A \parallel M$ or $M \parallel A$ but $P \parallel N$ or $N \parallel P$.

(56) If $A \parallel M$ and $M||X$, then $A||X$.

(57) If X is a plane, then $X||X$.

(58) If X is a plane and Y is a plane and $X||Y$, then $Y||X$.

(59) If X is a plane, then $X \neq \emptyset$.

(60) If $X||Y$ and $Y||Z$ and $Y \neq \emptyset$, then $X||Z$.

(61) If X is a plane and Y is a plane and Z is a plane but $X||Y$ and $Y||Z$ or $X||Y$ and $Z||Y$ or $Y||X$ and $Y||Z$ or $Y||X$ and $Z||Y$, then $X||Z$ and $Z||X$.

(62) If X is a plane and Y is a plane and $a \in X$ and $a \in Y$ and $X||Y$, then $X = Y$.

(63) If X is a plane and Y is a plane and Z is a plane and $X||Y$ and $X \neq Y$ and $a \in X \cap Z$ and $b \in X \cap Z$ and $c \in Y \cap Z$ and $d \in Y \cap Z$, then $a, b \parallel c, d$.

(64) Suppose X is a plane and Y is a plane and Z is a plane and $X||Y$ and $a \in X \cap Z$ and $b \in X \cap Z$ and $c \in Y \cap Z$ and $d \in Y \cap Z$ and $X \neq Y$ and $a \neq b$ and $c \neq d$. Then $X \cap Z \parallel Y \cap Z$.

(65) For all a, X such that X is a plane there exists Y such that $a \in Y$ and $X||Y$ and Y is a plane.

Let us consider A_1, a, X. Let us assume that X is a plane. The functor $a + X$ yields a subset of the points of A_1 and is defined as follows:

(Def.6) $a \in a + X$ and $X||a + X$ and $a + X$ is a plane.

Next we state four propositions:

(66) If X is a plane, then $a \in X$ if and only if $a + X = X$.

(67) If X is a plane, then $a + X = a + (q + X)$.

(68) If A is a line and X is a plane and $A||X$, then $a \cdot A \subseteq a + X$.

(69) If X is a plane and Y is a plane and $X||Y$, then $a+X=a+Y$.

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Graphs

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Summary. Definitions of graphs are introduced and their basic properties are proved. The following notions related to graph theory are introduced: Subgraph, Finite graph, Chain and oriented chain - as a finite sequence of edges, Path and oriented path - as a finite sequence of different edges, Cycle and oriented cycle, Incidency of graph's vertices, A sum of two graphs, A degree of a vertice, A set of all subgraphs of a graph. Many ideas in this article have been taken from [12].

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The terminology and notation used in this paper are introduced in the following papers: [10], [4], [5], [3], [9], [7], [6], [1], [8], [2], and [11]. We adopt the following convention: x, y, v will be arbitrary and n, m will be natural numbers. We consider multi graph structures which are systems

⟨vertices, edges, a source, a target⟩,

where the vertices, the edges constitute a set and the source, the target are a function from the edges into the vertices.

A multi graph structure is said to be a graph if:

(Def.1) the vertices of it is a non-empty set.

In the sequel G, G_1, G_2, G_3 are graphs. Let us consider G_1, G_2. Let us assume that the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2. The functor $G_1 \cup G_2$ yielding a graph is defined by the conditions (Def.2).

(Def.2) (i) The vertices of $G_1 \cup G_2 =$ (the vertices of G_1)$\cup$ the vertices of G_2,

(ii) the edges of $G_1 \cup G_2 =$ (the edges of G_1)$\cup$ the edges of G_2,

(iii) for every v such that $v \in$ the edges of G_1 holds (the source of $G_1 \cup G_2$)$(v) =$ (the source of G_1)(v) and (the target of $G_1 \cup G_2$)$(v) =$ (the target of G_1)(v),

(iv) for every v such that $v \in$ the edges of G_2 holds (the source of $G_1 \cup G_2$)$(v) =$ (the source of G_2)(v) and (the target of $G_1 \cup G_2$)$(v) =$ (the target of G_2)(v).

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Let G, G_1, G_2 be graphs. We say that G is a sum of G_1 and G_2 if and only if:

(Def.3) the target of $G_1 \approx$ the target of G_2 and the source of $G_1 \approx$ the source of G_2 and $G = G_1 \cup G_2$.

We now define five new attributes. A graph is oriented if:

(Def.4) for all x, y such that $x \in$ the edges of it and $y \in$ the edges of it and (the source of it)$(x) =$ (the source of it)(y) and (the target of it)$(x) =$ (the target of it)(y) holds $x = y$.

A graph is non-multi if it satisfies the condition (Def.5).

(Def.5) Given x, y. Suppose $x \in$ the edges of it and $y \in$ the edges of it but (the source of it)$(x) =$ (the source of it)(y) and (the target of it)$(x) =$ (the target of it)(y) or (the source of it)$(x) =$ (the target of it)(y) and (the source of it)$(y) =$ (the target of it)(x). Then $x = y$.

A graph is simple if:

(Def.6) for no x holds $x \in$ the edges of it and (the source of it)$(x) =$ (the target of it)(x).

A graph is connected if:

(Def.7) for no graphs G_1, G_2 holds (the vertices of G_1)$\cap$ the vertices of $G_2 = \emptyset$ and it is a sum of G_1 and G_2.

A multi graph structure is finite if:

(Def.8) the vertices of it is finite and the edges of it is finite.

In the sequel x, y will denote elements of the vertices of G. Let us consider G, x, y, v. We say that v joins x with y if and only if:

(Def.9) (the source of G)$(v) = x$ and (the target of G)$(v) = y$ or (the source of G)$(v) = y$ and (the target of G)$(v) = x$.

Let us consider G, and let x, y be elements of the vertices of G. We say that x and y are incydent if and only if:

(Def.10) there exists arbitrary v such that $v \in$ the edges of G and v joins x with y.

Let G be a graph. A finite sequence is called a chain of G if it satisfies the conditions (Def.11).

(Def.11) (i) For every n such that $1 \leq n$ and $n \leq \operatorname{len}$ it holds it$(n) \in$ the edges of G,

(ii) there exists a finite sequence p such that $\operatorname{len} p = \operatorname{len} \mathrm{it} + 1$ and for every n such that $1 \leq n$ and $n \leq \operatorname{len} p$ holds $p(n) \in$ the vertices of G and for every n such that $1 \leq n$ and $n \leq \operatorname{len}$ it there exist elements x', y' of the vertices of G such that $x' = p(n)$ and $y' = p(n+1)$ and it(n) joins x' with y'.

Let G be a graph. A chain of G is said to be an oriented chain of G if:

(Def.12) for every n such that $1 \leq n$ and $n < \operatorname{len}$ it holds (the source of G)(it$(n+1)$) $=$ (the target of G)(it(n)).

Let G be a graph. A chain of G is said to be a path of G if:

(Def.13) for all n, m such that $1 \leq n$ and $n < m$ and $m \leq \operatorname{len} \mathrm{it}$ holds $\mathrm{it}(n) \neq \mathrm{it}(m)$.

Let G be a graph. An oriented chain of G is said to be an oriented path of G if:

(Def.14) it is a path of G.

Let G be a graph. A path of G is said to be a cycle of G if it satisfies the condition (Def.15).

(Def.15) There exists a finite sequence p such that $\operatorname{len} p = \operatorname{len} \mathrm{it} + 1$ and for every n such that $1 \leq n$ and $n \leq \operatorname{len} p$ holds $p(n) \in$ the vertices of G and for every n such that $1 \leq n$ and $n \leq \operatorname{len} \mathrm{it}$ there exist elements x', y' of the vertices of G such that $x' = p(n)$ and $y' = p(n+1)$ and $\mathrm{it}(n)$ joins x' with y' and $p(1) = p(\operatorname{len} p)$.

Let G be a graph. An oriented path of G is called an oriented cycle of G if:

(Def.16) it is a cycle of G.

Let G be a graph. A graph is said to be a subgraph of G if it satisfies the conditions (Def.17).

(Def.17) (i) The vertices of it $\subseteq$ the vertices of G,

(ii) the edges of it $\subseteq$ the edges of G,

(iii) for every v such that $v \in$ the edges of it holds (the source of it)$(v) =$ (the source of G)(v) and (the target of it)$(v) =$ (the target of G)(v) and (the source of G)$(v) \in$ the vertices of it and (the target of G)$(v) \in$ the vertices of it.

We now define two new functors. Let G be an finite graph. The number of vertices of G yielding a natural number is defined by:

(Def.18) the number of vertices of $G =$ card (the vertices of G).

The number of edges of G yielding a natural number is defined by:

(Def.19) the number of edges of $G =$ card (the edges of G).

We now define two new functors. Let G be an finite graph, and let x be an element of the vertices of G. The functor $\mathrm{EdgIn}(x)$ yields a natural number and is defined as follows:

(Def.20) there exists a set X such that for an arbitrary z holds $z \in X$ if and only if $z \in$ the edges of G and (the target of G)$(z) = x$ and $\mathrm{EdgIn}(x) = \operatorname{card} X$.

The functor $\mathrm{EdgOut}(x)$ yielding a natural number is defined by:

(Def.21) there exists a set X such that for an arbitrary z holds $z \in X$ if and only if $z \in$ the edges of G and (the source of G)$(z) = x$ and $\mathrm{EdgOut}(x) = \operatorname{card} X$.

Let G be an finite graph, and let x be an element of the vertices of G. The degree of x yields a natural number and is defined by:

(Def.22) the degree of $x = \mathrm{EdgIn}(x) + \mathrm{EdgOut}(x)$.

Let G_1, G_2 be graphs. The predicate $G_1 \subseteq G_2$ is defined by:

(Def.23) G_1 is a subgraph of G_2.

Let G be a graph. The functor 2^G yields a set and is defined by:

(Def.24) for an arbitrary x holds $x \in 2^G$ if and only if x is a subgraph of G.

The scheme *GraphSeparation* deals with a graph $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a set X such that for an arbitrary x holds $x \in X$ if and only if x is a subgraph of $\mathcal{A}$ and $\mathcal{P}[x]$

for all values of the parameters.

Next we state a number of propositions:

(1) For every graph G holds dom (the source of G) = the edges of G and dom (the target of G) = the edges of G and rng (the source of G) $\subseteq$ the vertices of G and rng (the target of G) $\subseteq$ the vertices of G.

(2) For every element x of the vertices of G holds $x \in$ the vertices of G.

(3) For an arbitrary v such that $v \in$ the edges of G holds (the source of G)(v) $\in$ the vertices of G and (the target of G)(v) $\in$ the vertices of G.

(4) For every chain p of G holds $p \restriction \mathrm{Seg}\, n$ is a chain of G.

(5) If $G_1 \subseteq G$, then graph (the source of G_1) $\subseteq$ graph (the source of G) and graph (the target of G_1) $\subseteq$ graph (the target of G).

(6) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2, then graph (the source of $G_1 \cup G_2$) = graph (the source of G_1) $\cup$ graph (the source of G_2) and graph (the target of $G_1 \cup G_2$) = graph (the target of G_1) $\cup$ graph (the target of G_2).

(7) $G = G \cup G$.

(8) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2, then $G_1 \cup G_2 = G_2 \cup G_1$.

(9) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2 and the source of $G_1 \approx$ the source of G_3 and the target of $G_1 \approx$ the target of G_3 and the source of $G_2 \approx$ the source of G_3 and the target of $G_2 \approx$ the target of G_3, then $G_1 \cup G_2 \cup G_3 = G_1 \cup (G_2 \cup G_3)$.

(10) If G is a sum of G_1 and G_2, then G is a sum of G_2 and G_1.

(11) G is a sum of G and G.

(12) If there exists G such that $G_1 \subseteq G$ and $G_2 \subseteq G$, then $G_1 \cup G_2 = G_2 \cup G_1$.

(13) If there exists G such that $G_1 \subseteq G$ and $G_2 \subseteq G$ and $G_3 \subseteq G$, then $G_1 \cup G_2 \cup G_3 = G_1 \cup (G_2 \cup G_3)$.

(14) $G \subseteq G$.

(15) For all subgraphs H_1, H_2 of G such that the vertices of H_1 = the vertices of H_2 and the edges of H_1 = the edges of H_2 holds $H_1 = H_2$.

(16) If $G_1 \subseteq G_2$ and $G_2 \subseteq G_1$, then $G_1 = G_2$.

(17) If $G_1 \subseteq G_2$ and $G_2 \subseteq G_3$, then $G_1 \subseteq G_3$.

(18) If G is a sum of G_1 and G_2, then $G_1 \subseteq G$ and $G_2 \subseteq G$.

(19) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2, then $G_1 \subseteq G_1 \cup G_2$ and $G_2 \subseteq G_1 \cup G_2$.

(20) If there exists G such that $G_1 \subseteq G$ and $G_2 \subseteq G$, then $G_1 \subseteq G_1 \cup G_2$ and $G_2 \subseteq G_1 \cup G_2$.

(21) If $G_1 \subseteq G_3$ and $G_2 \subseteq G_3$ and G is a sum of G_1 and G_2, then $G \subseteq G_3$.

(22) If $G_1 \subseteq G$ and $G_2 \subseteq G$, then $G_1 \cup G_2 \subseteq G$.

(23) If $G_1 \subseteq G_2$, then $G_1 \cup G_2 = G_2$ and $G_2 \cup G_1 = G_2$.

(24) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2 but $G_1 \cup G_2 = G_2$ or $G_2 \cup G_1 = G_2$, then $G_1 \subseteq G_2$.

(25) If G_2 is a sum of G_1 and G_2 or G_2 is a sum of G_2 and G_1, then $G_1 \subseteq G_2$.

(26) If there exists G such that $G_1 \subseteq G$ and $G_2 \subseteq G$ but $G_2 = G_1 \cup G_2$ or $G_2 = G_2 \cup G_1$, then $G_1 \subseteq G_2$.

(27) For every oriented graph G such that $G_1 \subseteq G$ holds G_1 is oriented.

(28) For every non-multi graph G such that $G_1 \subseteq G$ holds G_1 is non-multi.

(29) For every simple graph G such that $G_1 \subseteq G$ holds G_1 is simple.

(30) $G_1 \in 2^G$ if and only if $G_1 \subseteq G$.

(31) $G \in 2^G$.

We now state several propositions:

(32) $G_1 \subseteq G_2$ if and only if $2^{G_1} \subseteq 2^{G_2}$.

(33) $2^G \neq \emptyset$.

(34) $\{G\} \subseteq 2^G$.

(35) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2 and $2^{G_1 \cup G_2} \subseteq 2^{G_1} \cup 2^{G_2}$, then $G_1 \subseteq G_2$ or $G_2 \subseteq G_1$.

(36) If the source of $G_1 \approx$ the source of G_2 and the target of $G_1 \approx$ the target of G_2, then $2^{G_1} \cup 2^{G_2} \subseteq 2^{G_1 \cup G_2}$.

(37) If $G_1 \in 2^G$ and $G_2 \in 2^G$, then $G_1 \cup G_2 \in 2^G$.

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Mostowski's Fundamental Operations - Part I

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Summary. In the chapter II.4 of his book [17] A.Mostowski introduces what he calls fundamental operations:
$A_1(a,b) = \{\{\langle 0,x\rangle,\langle 1,y\rangle\} : x \in y \wedge x \in a \wedge y \in a\}$,
$A_2(a,b) = \{a,b\}$,
$A_3(a,b) = \bigcup a$,
$A_4(a,b) = \{\{\langle x,y\rangle\} : x \in a \wedge y \in b\}$,
$A_5(a,b) = \{x \cup y : x \in a \wedge y \in b\}$,
$A_6(a,b) = \{x \setminus y : x \in a \wedge y \in b\}$,
$A_7(a,b) = \{x \circ y : x \in a \wedge y \in b\}$.
He proves that if a non-void class is closed under these operations then it is predicatively closed. Then he formulates sufficient criteria for a class to be a model of ZF set theory (theorem 4.12).

The article includes the translation of this part of Mostowski's book. The fundamental operations are defined (to be precise, not these operations, but the notions of closure of a class with respect to them). Some properties of classes closed under these operations are proved. At last it is proved that if a non-void class X is closed under the operations $A_1 - A_7$ then $D_H(a) \in X$ for every a in X and every H being formula of ZF language ($D_H(a)$ consists of all finite sequences with terms belonging to a which satisfy H in a).

MML Identifier: ZF_FUND1.

The articles [20], [12], [7], [10], [4], [11], [13], [18], [2], [1], [24], [19], [8], [5], [9], [6], [16], [21], [14], [22], [15], [3], and [23] provide the notation and terminology for this paper. For simplicity we follow the rules: V will be a universal class, a, b, x, y will be elements of V, X will be a subclass of V, o, p, q, r, s, t, u will be arbitrary, A, B will be sets, n will be an element of ω, f_1 will be a finite subset of ω, E will be a non-empty set, f will be a function from VAR into E, k will be a natural number, v_1, v_2 will be elements of VAR, and H, H' will be ZF-formulae. Let us consider A, B. The functor AB yielding a set is defined as follows:

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(Def.1) $p \in AB$ if and only if there exist q, r, s such that $p = \langle q, s\rangle$ and $\langle q, r\rangle \in A$ and $\langle r, s\rangle \in B$.

Let us consider V, x, y. Then xy is an element of V.

The function decode from ω into VAR is defined by:

(Def.2) for every p such that $p \in \omega$ holds $\mathrm{decode}(p) = x_{\mathrm{card}\, p}$.

Let us consider v_1. The functor ${}^{v_1}x$ yielding a natural number is defined by:

(Def.3) $x_{{}^{v_1}x} = v_1$.

Let A be a finite subset of VAR. The functor $\mathrm{code}(A)$ yielding a finite subset of ω is defined as follows:

(Def.4) $\mathrm{code}(A) = (\mathrm{decode}^{-1}) \circ A$.

Let us consider H. Then $\mathrm{Free}\, H$ is a finite subset of VAR.

Let us consider v_1. Then $\{v_1\}$ is a finite subset of VAR. Let us consider v_2. Then $\{v_1, v_2\}$ is a finite subset of VAR.

Let us consider H, E. The functor $\mathrm{D}_E(H)$ yielding a set is defined by:

(Def.5) $p \in \mathrm{D}_E(H)$ if and only if there exists f such that $p = (f \cdot \mathrm{decode}) \restriction \mathrm{code}(\mathrm{Free}\, H)$ and $f \in \mathrm{St}_E(H)$.

Let us consider n. Then $\{n\}$ is a finite subset of ω.

We now define several new predicates. Let us consider V, X. We say that X is closed w.r.t. A1 if and only if:

(Def.6) for every a such that $a \in X$ holds $\{\{\langle \mathbf{0}_V, x\rangle, \langle \mathbf{1}_V, y\rangle\} : x \in y \wedge x \in a \wedge y \in a\} \in X$.

We say that X is closed w.r.t. A2 if and only if:

(Def.7) for all a, b such that $a \in X$ and $b \in X$ holds $\{a, b\} \in X$.

We say that X is closed w.r.t. A3 if and only if:

(Def.8) for every a such that $a \in X$ holds $\bigcup a \in X$.

We say that X is closed w.r.t. A4 if and only if:

(Def.9) for all a, b such that $a \in X$ and $b \in X$ holds $\{\{\langle x, y\rangle\} : x \in a \wedge y \in b\} \in X$.

We say that X is closed w.r.t. A5 if and only if:

(Def.10) for all a, b such that $a \in X$ and $b \in X$ holds $\{x \cup y : x \in a \wedge y \in b\} \in X$.

We say that X is closed w.r.t. A6 if and only if:

(Def.11) for all a, b such that $a \in X$ and $b \in X$ holds $\{x \setminus y : x \in a \wedge y \in b\} \in X$.

We say that X is closed w.r.t. A7 if and only if:

(Def.12) for all a, b such that $a \in X$ and $b \in X$ holds $\{xy : x \in a \wedge y \in b\} \in X$.

Let us consider V, X. We say that X is closed w.r.t. A1-A7 if and only if:

(Def.13) X is closed w.r.t. A1 and X is closed w.r.t. A2 and X is closed w.r.t. A3 and X is closed w.r.t. A4 and X is closed w.r.t. A5 and X is closed w.r.t. A6 and X is closed w.r.t. A7.

We now state a number of propositions:

(1) $X \subseteq V$ but if $o \in X$, then o is an element of V but if $o \in A$ and $A \in X$, then o is an element of V.

(2) If X is closed w.r.t. A1-A7, then $o \in X$ if and only if $\{o\} \in X$ but if $A \in X$, then $\bigcup A \in X$.

(3) If X is closed w.r.t. A1-A7, then $\emptyset \in X$ and $\mathbf{0} \in X$.

(4) If X is closed w.r.t. A1-A7 and $A \in X$ and $B \in X$, then $A \cup B \in X$ and $A \setminus B \in X$ and $AB \in X$.

(5) If X is closed w.r.t. A1-A7 and $A \in X$ and $B \in X$, then $A \cap B \in X$.

(6) If X is closed w.r.t. A1-A7 and $o \in X$ and $p \in X$, then $\{o, p\} \in X$ and $\langle o, p\rangle \in X$.

(7) If X is closed w.r.t. A1-A7, then $\omega \subseteq X$.

(8) If X is closed w.r.t. A1-A7, then $\omega^{f_1} \subseteq X$.

(9) If X is closed w.r.t. A1-A7 and $a \in X$, then $a^{f_1} \in X$.

(10) If X is closed w.r.t. A1-A7 and $a \in \omega^{f_1}$ and $b \in X$, then $\{ax : x \in b\} \in X$.

(11) If X is closed w.r.t. A1-A7 and $n \in f_1$ and $a \in X$ and $b \in X$ and $b \subseteq a^{f_1}$, then $\{x : x \in a^{f_1 \setminus \{n\}} \wedge \bigvee_u \{\langle n, u\rangle\} \cup x \in b\} \in X$.

(12) If X is closed w.r.t. A1-A7 and $n \notin f_1$ and $a \in X$ and $b \in X$ and $b \subseteq a^{f_1}$, then $\{\{\langle n, x\rangle\} \cup y : x \in a \wedge y \in b\} \in X$.

(13) If X is closed w.r.t. A1-A7 and B is finite and for every o such that $o \in B$ holds $o \in X$, then $B \in X$.

(14) If X is closed w.r.t. A1-A7 and $A \subseteq X$ and $y \in A^{f_1}$, then $y \in X$.

(15) If X is closed w.r.t. A1-A7 and $n \notin f_1$ and $a \in X$ and $a \subseteq X$ and $y \in a^{f_1}$, then $\{\{\langle n, x\rangle\} \cup y : x \in a\} \in X$.

(16) Suppose X is closed w.r.t. A1-A7 and $n \notin f_1$ and $a \in X$ and $a \subseteq X$ and $y \in a^{f_1}$ and $b \subseteq a^{f_1 \cup \{n\}}$ and $b \in X$. Then $\{x : x \in a \wedge \{\langle n, x\rangle\} \cup y \in b\} \in X$.

(17) If X is closed w.r.t. A1-A7 and $a \in X$, then $\{\{\langle \mathbf{0}_V, x\rangle, \langle \mathbf{1}_V, x\rangle\} : x \in a\} \in X$.

(18) If X is closed w.r.t. A1-A7 and $E \in X$, then for all v_1, v_2 holds $\mathrm{D}_E(v_1{=}v_2) \in X$ and $\mathrm{D}_E(v_1 \epsilon v_2) \in X$.

(19) If X is closed w.r.t. A1-A7 and $E \in X$, then for every H such that $\mathrm{D}_E(H) \in X$ holds $\mathrm{D}_E(\neg H) \in X$.

(20) If X is closed w.r.t. A1-A7 and $E \in X$, then for all H, H' such that $\mathrm{D}_E(H) \in X$ and $\mathrm{D}_E(H') \in X$ holds $\mathrm{D}_E(H \wedge H') \in X$.

(21) If X is closed w.r.t. A1-A7 and $E \in X$, then for all H, v_1 such that $\mathrm{D}_E(H) \in X$ holds $\mathrm{D}_E(\forall_{v_1} H) \in X$.

(22) If X is closed w.r.t. A1-A7 and $E \in X$, then $\mathrm{D}_E(H) \in X$.

(23) If X is closed w.r.t. A1-A7, then $n \in X$ and $\mathbf{0}_V \in X$ and $\mathbf{1}_V \in X$.

(24) $\{\langle o, p\rangle, \langle p, p\rangle\}\{\langle p, q\rangle\} = \{\langle o, q\rangle, \langle p, q\rangle\}$.

(25) If $p \neq r$, then $\{\langle o, p\rangle, \langle q, r\rangle\}\{\langle p, s\rangle, \langle r, t\rangle\} = \{\langle o, s\rangle, \langle q, t\rangle\}$.

(26) ${}^{x_k}x = k$.

(27) $\mathrm{code}(\{v_1\}) = \{\mathrm{ord}({}^{v_1}x)\}$ and $\mathrm{code}(\{v_1, v_2\}) = \{\mathrm{ord}({}^{v_1}x), \mathrm{ord}({}^{v_2}x)\}$.

(28) $\mathrm{dom}\, f = \{o, q\}$ if and only if $\mathrm{graph}\, f = \{\langle o, f(o)\rangle, \langle q, f(q)\rangle\}$.

(29) $\mathrm{dom}\,\mathrm{decode} = \omega$ and $\mathrm{rng}\,\mathrm{decode} = \mathrm{VAR}$ and decode is one-to-one and decode^{-1} is one-to-one and $\mathrm{dom}(\mathrm{decode}^{-1}) = \mathrm{VAR}$ and $\mathrm{rng}(\mathrm{decode}^{-1}) = \omega$.

(30) For every finite subset A of VAR holds $A \approx \mathrm{code}(A)$.

(31) If $A \in \omega$, then $A = \mathrm{ord}(\mathrm{card}\, A)$ and $A = \mathrm{ord}({}^{x_{\mathrm{card}\, A}}x)$.

One can prove the following propositions:

(32) $\mathrm{dom}((f \cdot \mathrm{decode}) \restriction f_1) = f_1$ and $\mathrm{rng}((f \cdot \mathrm{decode}) \restriction f_1) \subseteq E$ and $(f \cdot \mathrm{decode}) \restriction f_1 \in E^{f_1}$ and $\mathrm{dom}(f \cdot \mathrm{decode}) = \omega$ and $\mathrm{rng}(f \cdot \mathrm{decode}) \subseteq E$.

(33) $\mathrm{decode}(\mathrm{ord}({}^{v_1}x)) = v_1$ and $\mathrm{decode}^{-1}(v_1) = \mathrm{ord}({}^{v_1}x)$ and $(f \cdot \mathrm{decode})(\mathrm{ord}({}^{v_1}x)) = f(v_1)$.

(34) For every finite subset A of VAR holds $p \in \mathrm{code}(A)$ if and only if there exists v_1 such that $v_1 \in A$ and $p = \mathrm{ord}({}^{v_1}x)$.

(35) For all finite subsets A, B of VAR holds $\mathrm{code}(A \cup B) = \mathrm{code}(A) \cup \mathrm{code}(B)$ and $\mathrm{code}(A \setminus B) = \mathrm{code}(A) \setminus \mathrm{code}(B)$.

(36) If $v_1 \in \mathrm{Free}\, H$, then $((f \cdot \mathrm{decode}) \restriction \mathrm{code}(\mathrm{Free}\, H))(\mathrm{ord}({}^{v_1}x)) = f(v_1)$.

(37) For all functions f, g from VAR into E such that $(f \cdot \mathrm{decode}) \restriction \mathrm{code}(\mathrm{Free}\, H) = (g \cdot \mathrm{decode}) \restriction \mathrm{code}(\mathrm{Free}\, H)$ and $f \in \mathrm{St}_E(H)$ holds $g \in \mathrm{St}_E(H)$.

(38) If $p \in E^{f_1}$, then there exists f such that $p = (f \cdot \mathrm{decode}) \restriction f_1$.

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A Projective Closure and Projective Horizon of an Affine Space

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Summary. With every affine space A we correlate two incidence structures. The first, called Inc-ProjSp(A), is the usual projective closure of A, i.e. the structure obtained from A by adding directions of lines and planes of A. The second, called projective horizon of A, is the structure build from directions. We prove that Inc-ProjSp(A) is always a projective space, and projective horizon of A is a projective space provided A is at least 3-dimensional. Some evident relationships between projective and affine configurational axioms that may hold in A and in Inc-ProjSp(A) are established.

MML Identifier: AFPROJ.

The notation and terminology used in this paper have been introduced in the following articles: [9], [11], [12], [8], [6], [13], [10], [3], [4], [5], [1], [7], and [2]. We adopt the following rules: A_1 will denote an affine space, A, K, M, X, Y will denote subsets of the points of A_1, and x, y will be arbitrary. Next we state several propositions:

(1) If A_1 is an affine plane and $X =$ the points of A_1, then X is a plane.

(2) If A_1 is an affine plane and X is a plane, then $X =$ the points of A_1.

(3) If A_1 is an affine plane and X is a plane and Y is a plane, then $X = Y$.

(4) If $X =$ the points of A_1 and X is a plane, then A_1 is an affine plane.

(5) If $A \nparallel K$ and $A||X$ and $A||Y$ and $K||X$ and $K||Y$ and A is a line and K is a line and X is a plane and Y is a plane, then $X||Y$.

(6) If A is a line and X is a plane and Y is a plane and $A||X$ and $X||Y$, then $A||Y$.

Let D be a non-empty set, and let X be a set. Then $D \cup X$ is a non-empty set.

Let us consider A_1. The lines of A_1 yields a family of subsets of the points of A_1 and is defined as follows:

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(Def.1) the lines of $A_1 = \{A : A$ is a line $\}$.

Let us consider A_1. The planes of A_1 yielding a family of subsets of the points of A_1 is defined as follows:

(Def.2) the planes of $A_1 = \{A : A$ is a plane $\}$.

The following two propositions are true:

(7) For every x holds $x \in$ the lines of A_1 if and only if there exists X such that $x = X$ and X is a line.

(8) For every x holds $x \in$ the planes of A_1 if and only if there exists X such that $x = X$ and X is a plane.

Let us consider A_1. The parallelity of lines of A_1 yields an equivalence relation of the lines of A_1 and is defined by:

(Def.3) the parallelity of lines of $A_1 = \{\langle K, M\rangle : K$ is a line $\wedge$ M is a line $\wedge K||M\}$.

Let us consider A_1. The parallelity of planes of A_1 yielding an equivalence relation of the planes of A_1 is defined as follows:

(Def.4) the parallelity of planes of $A_1 = \{\langle X, Y\rangle : X$ is a plane $\wedge$ Y is a plane $\wedge X||Y\}$.

Let us consider A_1, X. Let us assume that X is a line. The direction of X yields a subset of the lines of A_1 and is defined by:

(Def.5) the direction of $X = [X]_{\text{the parallelity of lines of } A_1}$.

Let us consider A_1, X. Let us assume that X is a plane. The direction of X yielding a subset of the planes of A_1 is defined as follows:

(Def.6) the direction of $X = [X]_{\text{the parallelity of planes of } A_1}$.

Next we state several propositions:

(9) If X is a line, then for every x holds $x \in$ the direction of X if and only if there exists Y such that $x = Y$ and Y is a line and $X||Y$.

(10) If X is a plane, then for every x holds $x \in$ the direction of X if and only if there exists Y such that $x = Y$ and Y is a plane and $X||Y$.

(11) If X is a line and Y is a line, then the direction of $X =$ the direction of Y if and only if $X \parallel Y$.

(12) If X is a line and Y is a line, then the direction of $X =$ the direction of Y if and only if $X||Y$.

(13) If X is a plane and Y is a plane, then
the direction of $X =$ the direction of Y
if and only if $X||Y$.

Let us consider A_1. The directions of lines of A_1 yields a non-empty set and is defined as follows:

(Def.7) the directions of lines of $A_1 =$ Classes(the parallelity of lines of A_1).

Let us consider A_1. The directions of planes of A_1 yielding a non-empty set is defined by:

(Def.8) the directions of planes of A_1 = Classes(the parallelity of planes of A_1).

One can prove the following propositions:

(14) For every x holds $x \in$ the directions of lines of A_1 if and only if there exists X such that $x =$ the direction of X and X is a line.

(15) For every x holds $x \in$ the directions of planes of A_1 if and only if there exists X such that $x =$ the direction of X and X is a plane.

(16) (the points of A_1) $\cap$ the directions of lines of $A_1 = \emptyset$.

(17) If A_1 is an affine plane, then (the lines of A_1) $\cap$ the directions of planes of $A_1 = \emptyset$.

(18) For every x holds $x \in [\!:$ the lines of A_1, $\{1\}\,:\!]$ if and only if there exists X such that $x = \langle X, 1\rangle$ and X is a line.

(19) For every x holds $x \in [\!:$ the directions of planes of A_1, $\{2\}\,:\!]$ if and only if there exists X such that $x = \langle$ the direction of $X, 2\rangle$ and X is a plane.

Let us consider A_1. The projective points over A_1 yielding a non-empty set is defined as follows:

(Def.9) the projective points over A_1 = (the points of A_1) $\cup$ the directions of lines of A_1.

Let us consider A_1. The functor $L(A_1)$ yielding a non-empty set is defined as follows:

(Def.10) $L(A_1) = [\!:$ the lines of A_1, $\{1\}\,:\!] \cup [\!:$ the directions of planes of A_1, $\{2\}\,:\!]$.

Let us consider A_1. The functor $\mathbf{I}_{A_1}$ yielding a relation between the projective points over A_1 and $L(A_1)$ is defined by the condition (Def.11).

(Def.11) Given x, y. Then $\langle x, y\rangle \in \mathbf{I}_{A_1}$ if and only if there exists K such that K is a line and $y = \langle K, 1\rangle$ but $x \in$ the points of A_1 and $x \in K$ or $x =$ the direction of K or there exist K, X such that K is a line and X is a plane and $x =$ the direction of K and $y = \langle$ the direction of $X, 2\rangle$ and $K || X$.

Let us consider A_1. The incidence of directions of A_1 yields a relation between the directions of lines of A_1 and the directions of planes of A_1 and is defined as follows:

(Def.12) for all x, y holds $\langle x, y\rangle \in$ the incidence of directions of A_1 if and only if there exist A, X such that $x =$ the direction of A and $y =$ the direction of X and A is a line and X is a plane and $A || X$.

Let us consider A_1. The functor Inc-ProjSp(A_1) yielding a projective incidence structure is defined as follows:

(Def.13) Inc-ProjSp(A_1) = $\langle$the projective points over A_1, $L(A_1), \mathbf{I}_{A_1}\rangle$.

Let us consider A_1. The projective horizon of A_1 yielding a projective incidence structure is defined as follows:

(Def.14) the projective horizon of A_1 = $\langle$the directions of lines of A_1, the directions of planes of A_1, the incidence of directions of $A_1\rangle$.

We now state several propositions:

(20) For every x holds x is an element of the points of Inc-ProjSp(A_1) if and only if x is an element of the points of A_1 or there exists X such that $x =$ the direction of X and X is a line.

(21) x is an element of the points of the projective horizon of A_1 if and only if there exists X such that $x =$ the direction of X and X is a line.

(22) If x is an element of the points of the projective horizon of A_1, then x is an element of the points of Inc-ProjSp(A_1).

(23) For every x holds x is an element of the lines of Inc-ProjSp(A_1) if and only if there exists X such that $x = \langle X, 1\rangle$ and X is a line or $x = \langle$ the direction of $X, 2\rangle$ and X is a plane.

(24) x is an element of the lines of the projective horizon of A_1 if and only if there exists X such that $x =$ the direction of X and X is a plane.

(25) If x is an element of the lines of the projective horizon of A_1, then $\langle x, 2\rangle$ is an element of the lines of Inc-ProjSp(A_1).

For simplicity we adopt the following rules: x will denote an element of the points of A_1, X, Y, X' will denote subsets of the points of A_1, a, p, q will denote elements of the points of Inc-ProjSp(A_1), and A will denote an element of the lines of Inc-ProjSp(A_1). We now state a number of propositions:

(26) If $x = a$ and $\langle X, 1\rangle = A$, then $a \mid A$ if and only if X is a line and $x \in X$.

(27) If $x = a$ and $\langle$ the direction of $X, 2\rangle = A$ and X is a plane, then $a \nmid A$.

(28) If $a =$ the direction of Y and $\langle X, 1\rangle = A$ and Y is a line and X is a line, then $a \mid A$ if and only if $Y || X$.

(29) If $a =$ the direction of Y and $A = \langle$ the direction of $X, 2\rangle$ and Y is a line and X is a plane, then $a \mid A$ if and only if $Y || X$.

(30) If X is a line and $a =$ the direction of X and $A = \langle X, 1\rangle$, then $a \mid A$.

(31) If X is a line and Y is a plane and $X \subseteq Y$ and $a =$ the direction of X and $A = \langle$ the direction of $Y, 2\rangle$, then $a \mid A$.

(32) If Y is a plane and $X \subseteq Y$ and $X' \parallel X$ and $a =$ the direction of X' and $A = \langle$ the direction of $Y, 2\rangle$, then $a \mid A$.

(33) If $A = \langle$ the direction of $X, 2\rangle$ and X is a plane and $a \mid A$, then a is not an element of the points of A_1.

(34) If $A = \langle X, 1\rangle$ and X is a line and $p \mid A$ and p is not an element of the points of A_1, then $p =$ the direction of X.

(35) If $A = \langle X, 1\rangle$ and X is a line and $p \mid A$ and $a \mid A$ and $a \neq p$ and p is not an element of the points of A_1, then a is an element of the points of A_1.

(36) For every element a of the points of the projective horizon of A_1 and for every element A of the lines of the projective horizon of A_1 such that $a =$ the direction of X and $A =$ the direction of Y and X is a line and Y is a plane holds $a \mid A$ if and only if $X || Y$.

(37) For every element a of the points of the projective horizon of A_1 and for every element a' of the points of Inc-ProjSp(A_1) and for every element A of the lines of the projective horizon of A_1 and for every element A' of the lines of Inc-ProjSp(A_1) such that $a' = a$ and $A' = \langle A, 2 \rangle$ holds $a \mid A$ if and only if $a' \mid A'$.

In the sequel P, Q denote elements of the lines of Inc-ProjSp(A_1). We now state several propositions:

(38) For all elements a, b of the points of the projective horizon of A_1 and for all elements A, K of the lines of the projective horizon of A_1 such that $a \mid A$ and $a \mid K$ and $b \mid A$ and $b \mid K$ holds $a = b$ or $A = K$.

(39) For every element A of the lines of the projective horizon of A_1 there exist elements a, b, c of the points of the projective horizon of A_1 such that $a \mid A$ and $b \mid A$ and $c \mid A$ and $a \neq b$ and $b \neq c$ and $c \neq a$.

(40) For every elements a, b of the points of the projective horizon of A_1 there exists an element A of the lines of the projective horizon of A_1 such that $a \mid A$ and $b \mid A$.

(41) For all elements x, y of the points of the projective horizon of A_1 and for every element X of the lines of Inc-ProjSp(A_1) such that $x \neq y$ and $\langle x, X \rangle \in$ the incidence of Inc-ProjSp(A_1) and $\langle y, X \rangle \in$ the incidence of Inc-ProjSp(A_1) there exists an element Y of the lines of the projective horizon of A_1 such that $X = \langle Y, 2 \rangle$.

(42) For every element x of the points of Inc-ProjSp(A_1) and for every element X of the lines of the projective horizon of A_1 such that $\langle x, \langle X, 2 \rangle \rangle \in$ the incidence of Inc-ProjSp(A_1) holds x is an element of the points of the projective horizon of A_1.

(43) If Y is a plane and X is a line and X' is a line and $X \subseteq Y$ and $X' \subseteq Y$ and $P = \langle X, 1 \rangle$ and $Q = \langle X', 1 \rangle$, then there exists q such that $q \mid P$ and $q \mid Q$.

(44) Let a, b, c, d, p be elements of the points of the projective horizon of A_1. Let M, N, P, Q be elements of the lines of the projective horizon of A_1. Suppose that

(i) $a \mid M$,
(ii) $b \mid M$,
(iii) $c \mid N$,
(iv) $d \mid N$,
(v) $p \mid M$,
(vi) $p \mid N$,
(vii) $a \mid P$,
(viii) $c \mid P$,
(ix) $b \mid Q$,
(x) $d \mid Q$,
(xi) $p \nmid P$,
(xii) $p \nmid Q$,

(xiii) $M \neq N$.

Then there exists an element q of the points of the projective horizon of A_1 such that $q \mid P$ and $q \mid Q$.

Let us consider A_1. Then Inc-ProjSp(A_1) is a projective space defined in terms of incidence.

Let A_1 be an affine plane. Then Inc-ProjSp(A_1) is a 2-dimensional projective space defined in terms of incidence.

The following propositions are true:

(45) If Inc-ProjSp(A_1) is 2-dimensional, then A_1 is an affine plane.

(46) If A_1 is not an affine plane, then the projective horizon of A_1 is a projective space defined in terms of incidence.

(47) If the projective horizon of A_1 is a projective space defined in terms of incidence, then A_1 is not an affine plane.

(48) Let M, N be subsets of the points of A_1. Let o, a, b, c, a', b', c' be elements of the points of A_1. Suppose that

(i) M is a line,

(ii) N is a line,

(iii) $M \neq N$,

(iv) $o \in M$,

(v) $o \in N$,

(vi) $o \neq a$,

(vii) $o \neq a'$,

(viii) $o \neq b$,

(ix) $o \neq b'$,

(x) $o \neq c$,

(xi) $o \neq c'$,

(xii) $a \in M$,

(xiii) $b \in M$,

(xiv) $c \in M$,

(xv) $a' \in N$,

(xvi) $b' \in N$,

(xvii) $c' \in N$,

(xviii) $a, b' \parallel b, a'$,

(xix) $b, c' \parallel c, b'$,

(xx) $a = b$ or $b = c$ or $a = c$.

Then $a, c' \parallel c, a'$.

(49) If Inc-ProjSp(A_1) is Pappian, then A_1 is Pappian.

(50) Let A, P, C be subsets of the points of A_1. Let o, a, b, c, a', b', c' be elements of the points of A_1. Suppose that

(i) $o \in A$,

(ii) $o \in P$,

(iii) $o \in C$,

(iv) $o \neq a$,

(v) $o \neq b$,
(vi) $o \neq c$,
(vii) $a \in A$,
(viii) $a' \in A$,
(ix) $b \in P$,
(x) $b' \in P$,
(xi) $c \in C$,
(xii) $c' \in C$,
(xiii) A is a line,
(xiv) P is a line,
(xv) C is a line,
(xvi) $A \neq P$,
(xvii) $A \neq C$,
(xviii) $a, b \parallel a', b'$,
(xix) $a, c \parallel a', c'$,
(xx) $o = a'$ or $a = a'$.

Then $b, c \parallel b', c'$.

(51) If Inc-ProjSp(A_1) is Desarguesian, then A_1 is Desarguesian.

(52) If Inc-ProjSp(A_1) is Fanoian, then A_1 is Fanoian.

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Schemes [1]

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Summary. Some basic schemes of quantifier calculus are proved.

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In the sequel a, b will be arbitrary. In this article we present several logical schemes. The scheme *Schemat0* concerns a unary predicate $\mathcal{P}$, and states that:

there exists a such that $\mathcal{P}[a]$

provided the parameter meets the following requirement:

- for every a holds $\mathcal{P}[a]$.

The scheme *Schemat1a* deals with $\mathcal{Q}$, and a unary predicate $\mathcal{P}$, and states that:

for every a holds $\mathcal{P}[a]$ and $\mathcal{Q}[]$

provided the parameters meet the following requirement:

- for every a holds $\mathcal{P}[a]$ and $\mathcal{Q}[]$.

The scheme *Schemat1b* concerns $\mathcal{Q}$, and a unary predicate $\mathcal{P}$, and states that:

for every a holds $\mathcal{P}[a]$ and $\mathcal{Q}[]$

provided the parameters have the following property:

- for every a holds $\mathcal{P}[a]$ and $\mathcal{Q}[]$.

The scheme *Schemat2a* concerns $\mathcal{Q}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a such that $\mathcal{P}[a]$ or $\mathcal{Q}[]$

provided the parameters meet the following requirement:

- there exists a such that $\mathcal{P}[a]$ or $\mathcal{Q}[]$.

The scheme *Schemat2b* deals with $\mathcal{Q}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a such that $\mathcal{P}[a]$ or $\mathcal{Q}[]$

provided the following condition is met:

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- there exists a such that $\mathcal{P}[a]$ or $\mathcal{Q}[]$.

The scheme *Schemat3* concerns a binary predicate $\mathcal{P}$, and states that:

for every b there exists a such that $\mathcal{P}[a, b]$

provided the parameter has the following property:

- there exists a such that for every b holds $\mathcal{P}[a, b]$.

The scheme *Schemat4a* concerns two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists a such that $\mathcal{P}[a]$ or there exists a such that $\mathcal{Q}[a]$

provided the following condition is satisfied:

- there exists a such that $\mathcal{P}[a]$ or $\mathcal{Q}[a]$.

The scheme *Schemat4b* deals with two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists a such that $\mathcal{P}[a]$ or $\mathcal{Q}[a]$

provided the parameters meet the following requirement:

- there exists a such that $\mathcal{P}[a]$ or there exists a such that $\mathcal{Q}[a]$.

The scheme *Schemat5* concerns two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists a such that $\mathcal{P}[a]$ and there exists a such that $\mathcal{Q}[a]$

provided the following condition is met:

- there exists a such that $\mathcal{P}[a]$ and $\mathcal{Q}[a]$.

The scheme *Schemat6a* concerns two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

for every a holds $\mathcal{P}[a]$ and for every a holds $\mathcal{Q}[a]$

provided the parameters satisfy the following condition:

- for every a holds $\mathcal{P}[a]$ and $\mathcal{Q}[a]$.

The scheme *Schemat6b* deals with two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

for every a holds $\mathcal{P}[a]$ and $\mathcal{Q}[a]$

provided the following requirement is met:

- for every a holds $\mathcal{P}[a]$ and for every a holds $\mathcal{Q}[a]$.

The scheme *Schemat7* deals with two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

for every a holds $\mathcal{P}[a]$ or $\mathcal{Q}[a]$

provided the following condition is satisfied:

- for every a holds $\mathcal{P}[a]$ or for every a holds $\mathcal{Q}[a]$.

The scheme *Schemat8* concerns two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

if for every a holds $\mathcal{P}[a]$, then for every a holds $\mathcal{Q}[a]$

provided the parameters satisfy the following condition:

- for every a such that $\mathcal{P}[a]$ holds $\mathcal{Q}[a]$.

The scheme *Schemat9* concerns two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

for every a holds $\mathcal{P}[a]$ if and only if for every a holds $\mathcal{Q}[a]$

provided the parameters have the following property:

- for every a holds $\mathcal{P}[a]$ if and only if $\mathcal{Q}[a]$.

The scheme *Schemat10a* concerns $\mathcal{P}$ and states that:

$\mathcal{P}[]$

provided the parameter satisfies the following condition:

- for every a holds $\mathcal{P}[]$.

The scheme *Schemat10b* concerns $\mathcal{P}$ and states that:

for every a holds $\mathcal{P}[]$

provided the parameter satisfies the following condition:

- $\mathcal{P}[]$.

The scheme *Schemat11a* concerns $\mathcal{Q}$, and a unary predicate $\mathcal{P}$, and states that:

for every a holds $\mathcal{P}[a]$ or $\mathcal{Q}[]$

provided the following requirement is met:

- for every a holds $\mathcal{P}[a]$ or $\mathcal{Q}[]$.

The scheme *Schemat11b* deals with $\mathcal{Q}$, and a unary predicate $\mathcal{P}$, and states that:

for every a holds $\mathcal{P}[a]$ or $\mathcal{Q}[]$

provided the parameters satisfy the following condition:

- for every a holds $\mathcal{P}[a]$ or $\mathcal{Q}[]$.

The scheme *Schemat12a* concerns $\mathcal{Q}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a such that $\mathcal{Q}[]$ and $\mathcal{P}[a]$

provided the following condition is satisfied:

- $\mathcal{Q}[]$ and there exists a such that $\mathcal{P}[a]$.

The scheme *Schemat12b* concerns $\mathcal{Q}$, and a unary predicate $\mathcal{P}$, and states that:

$\mathcal{Q}[]$ and there exists a such that $\mathcal{P}[a]$

provided the following condition is satisfied:

- there exists a such that $\mathcal{Q}[]$ and $\mathcal{P}[a]$.

The scheme *Schemat13a* concerns $\mathcal{Q}$, and a unary predicate $\mathcal{P}$, and states that:

for every a such that $\mathcal{Q}[]$ holds $\mathcal{P}[a]$

provided the parameters satisfy the following condition:

- if $\mathcal{Q}[]$, then for every a holds $\mathcal{P}[a]$.

The scheme *Schemat13b* deals with $\mathcal{Q}$, and a unary predicate $\mathcal{P}$, and states that:

if $\mathcal{Q}[]$, then for every a holds $\mathcal{P}[a]$

provided the parameters satisfy the following condition:

- for every a such that $\mathcal{Q}[]$ holds $\mathcal{P}[a]$.

The scheme *Schemat14* concerns $\mathcal{Q}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a such that if $\mathcal{Q}[]$, then $\mathcal{P}[a]$

provided the parameters meet the following requirement:

- if $\mathcal{Q}[]$, then there exists a such that $\mathcal{P}[a]$.

The scheme *Schemat15* deals with $\mathcal{Q}$, and a unary predicate $\mathcal{P}$, and states that:

for every a such that $\mathcal{P}[a]$ holds $\mathcal{Q}[]$
provided the following condition is met:

- if there exists a such that $\mathcal{P}[a]$, then $\mathcal{Q}[]$.

The scheme *Schemat16* deals with $\mathcal{Q}$, and a unary predicate $\mathcal{P}$, and states that:

there exists a such that if $\mathcal{P}[a]$, then $\mathcal{Q}[]$
provided the parameters meet the following requirement:

- if for every a holds $\mathcal{P}[a]$, then $\mathcal{Q}[]$.

The scheme *Schemat17* concerns $\mathcal{Q}$, and a unary predicate $\mathcal{P}$, and states that:

if for every a holds $\mathcal{P}[a]$, then $\mathcal{Q}[]$
provided the parameters meet the following requirement:

- for every a such that $\mathcal{P}[a]$ holds $\mathcal{Q}[]$.

The scheme *Schemat18a* deals with two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists a such that for every b holds $\mathcal{P}[a]$ or $\mathcal{Q}[b]$
provided the following condition is satisfied:

- there exists a such that $\mathcal{P}[a]$ or for every b holds $\mathcal{Q}[b]$.

The scheme *Schemat18b* deals with two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists a such that $\mathcal{P}[a]$ or for every b holds $\mathcal{Q}[b]$
provided the parameters meet the following condition:

- there exists a such that for every b holds $\mathcal{P}[a]$ or $\mathcal{Q}[b]$.

The scheme *Schemat19a* concerns two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

for every b there exists a such that $\mathcal{P}[a]$ or $\mathcal{Q}[b]$
provided the following condition is met:

- there exists a such that $\mathcal{P}[a]$ or for every b holds $\mathcal{Q}[b]$.

The scheme *Schemat19b* concerns two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists a such that $\mathcal{P}[a]$ or for every b holds $\mathcal{Q}[b]$
provided the following condition is met:

- for every b there exists a such that $\mathcal{P}[a]$ or $\mathcal{Q}[b]$.

The scheme *Schemat20a* deals with two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

for every b there exists a such that $\mathcal{P}[a]$ or $\mathcal{Q}[b]$
provided the following condition is met:

- there exists a such that for every b holds $\mathcal{P}[a]$ or $\mathcal{Q}[b]$.

The scheme *Schemat20b* concerns two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists a such that for every b holds $\mathcal{P}[a]$ or $\mathcal{Q}[b]$
provided the following requirement is met:

- for every b there exists a such that $\mathcal{P}[a]$ or $\mathcal{Q}[b]$.

The scheme *Schemat21a* deals with two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists a such that for every b holds $\mathcal{P}[a]$ and $\mathcal{Q}[b]$
provided the following condition is satisfied:

- there exists a such that $\mathcal{P}[a]$ and for every b holds $\mathcal{Q}[b]$.

The scheme *Schemat21b* deals with two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists a such that $\mathcal{P}[a]$ and for every b holds $\mathcal{Q}[b]$
provided the following condition is satisfied:

- there exists a such that for every b holds $\mathcal{P}[a]$ and $\mathcal{Q}[b]$.

The scheme *Schemat22a* deals with two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

for every b there exists a such that $\mathcal{P}[a]$ and $\mathcal{Q}[b]$
provided the parameters meet the following condition:

- there exists a such that $\mathcal{P}[a]$ and for every b holds $\mathcal{Q}[b]$.

The scheme *Schemat22b* deals with two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists a such that $\mathcal{P}[a]$ and for every b holds $\mathcal{Q}[b]$
provided the following requirement is met:

- for every b there exists a such that $\mathcal{P}[a]$ and $\mathcal{Q}[b]$.

The scheme *Schemat23a* deals with two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

for every b there exists a such that $\mathcal{P}[a]$ and $\mathcal{Q}[b]$
provided the following requirement is met:

- there exists a such that for every b holds $\mathcal{P}[a]$ and $\mathcal{Q}[b]$.

The scheme *Schemat23b* deals with two unary predicates $\mathcal{P}$ and $\mathcal{Q}$, and states that:

there exists a such that for every b holds $\mathcal{P}[a]$ and $\mathcal{Q}[b]$
provided the parameters satisfy the following condition:

- for every b there exists a such that $\mathcal{P}[a]$ and $\mathcal{Q}[b]$.

The scheme *Schemat24a* concerns a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

for every a there exists b such that if $\mathcal{P}[a,b]$, then $\mathcal{Q}[a]$
provided the parameters satisfy the following condition:

- for every a such that for every b holds $\mathcal{P}[a,b]$ holds $\mathcal{Q}[a]$.

The scheme *Schemat24b* deals with a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

for every a such that for every b holds $\mathcal{P}[a,b]$ holds $\mathcal{Q}[a]$
provided the following requirement is met:

- for every a there exists b such that if $\mathcal{P}[a,b]$, then $\mathcal{Q}[a]$.

The scheme *Schemat25a* concerns a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

for all a, b such that $\mathcal{P}[a,b]$ holds $\mathcal{Q}[a]$
provided the parameters have the following property:

- for every a such that there exists b such that $\mathcal{P}[a,b]$ holds $\mathcal{Q}[a]$.

The scheme *Schemat25b* concerns a unary predicate $\mathcal{Q}$, and a binary predicate $\mathcal{P}$, and states that:

for every a such that there exists b such that $\mathcal{P}[a,b]$ holds $\mathcal{Q}[a]$
provided the following condition is met:

- for all a, b such that $\mathcal{P}[a,b]$ holds $\mathcal{Q}[a]$.

The scheme *Schemat26* deals with a binary predicate $\mathcal{P}$, and states that:
there exists a such that for every b holds $\mathcal{P}[a,b]$
provided the following condition is met:

- for all a, b holds $\mathcal{P}[a,b]$.

The scheme *Schemat27* deals with a binary predicate $\mathcal{P}$, and states that:
for every a holds $\mathcal{P}[a,a]$
provided the parameter meets the following condition:

- for all a, b holds $\mathcal{P}[a,b]$.

The scheme *Schemat28* concerns a binary predicate $\mathcal{P}$, and states that:
there exists b such that for every a holds $\mathcal{P}[a,b]$
provided the following requirement is met:

- for all a, b holds $\mathcal{P}[a,b]$.

The scheme *Schemat29* deals with a binary predicate $\mathcal{P}$, and states that:
for every b there exists a such that $\mathcal{P}[a,b]$
provided the parameter has the following property:

- there exists a such that for every b holds $\mathcal{P}[a,b]$.

The scheme *Schemat30* deals with a binary predicate $\mathcal{P}$, and states that:
there exists a such that $\mathcal{P}[a,a]$
provided the parameter meets the following requirement:

- there exists a such that for every b holds $\mathcal{P}[a,b]$.

The scheme *Schemat31* concerns a binary predicate $\mathcal{P}$, and states that:
for every a there exists b such that $\mathcal{P}[b,a]$
provided the following condition is satisfied:

- for every a holds $\mathcal{P}[a,a]$.

The scheme *Schemat32* concerns a binary predicate $\mathcal{P}$, and states that:
there exists a such that $\mathcal{P}[a,a]$
provided the parameter meets the following condition:

- for every a holds $\mathcal{P}[a,a]$.

The scheme *Schemat33* deals with a binary predicate $\mathcal{P}$, and states that:
for every a there exists b such that $\mathcal{P}[a,b]$
provided the following condition is satisfied:

- for every a holds $\mathcal{P}[a,a]$.

The scheme *Schemat34* concerns a binary predicate $\mathcal{P}$, and states that:
there exists b such that $\mathcal{P}[b,b]$
provided the parameter meets the following requirement:

- there exists b such that for every a holds $\mathcal{P}[a,b]$.

The scheme *Schemat35* deals with a binary predicate $\mathcal{P}$, and states that:
for every a there exists b such that $\mathcal{P}[a,b]$
provided the parameter meets the following condition:

- there exists b such that for every a holds $\mathcal{P}[a,b]$.

The scheme *Schemat36* deals with a binary predicate $\mathcal{P}$, and states that:
there exist a, b such that $\mathcal{P}[a,b]$

provided the parameter meets the following requirement:

- for every b there exists a such that $\mathcal{P}[a, b]$.

The scheme *Schemat37* deals with a binary predicate $\mathcal{P}$, and states that:

there exist a, b such that $\mathcal{P}[a, b]$

provided the following condition is satisfied:

- there exists a such that $\mathcal{P}[a, a]$.

The scheme *Schemat38* concerns a binary predicate $\mathcal{P}$, and states that:

there exist a, b such that $\mathcal{P}[a, b]$

provided the parameter satisfies the following condition:

- for every a there exists b such that $\mathcal{P}[a, b]$.

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Algebra of Normal Forms Is a Heyting Algebra [1]

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Summary. We prove that the lattice of normal forms over an arbitrary set, introduced in [7], is an implicative lattice. The relative psedo-complement $\alpha \Rightarrow \beta$ is defined as $\bigsqcup_{\alpha_1 \cup \alpha_2 = \alpha} -\alpha_1 \sqcap \alpha_2 \rightarrowtail \beta$, where $-\alpha$ is the pseudo-complement of α and $\alpha \rightarrowtail \beta$ is a rather strong implication introduced in this paper.

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The articles [13], [4], [5], [2], [14], [3], [8], [6], [15], [9], [16], [10], [11], [12], [7], and [1] provide the notation and terminology for this paper. One can prove the following proposition

(1) For all non-empty sets A, B, C and for every function f from A into B such that for every element x of A holds $f(x) \in C$ holds f is a function from A into C.

In the sequel A will be a non-empty set and a will be an element of A. Let us consider A, and let B, C be elements of $\mathrm{Fin}\,A$. Let us note that one can characterize the predicate $B \subseteq C$ by the following (equivalent) condition:

(Def.1) for every a such that $a \in B$ holds $a \in C$.

Let A be a non-empty set, and let B be a non-empty subset of A. Then $\overset{B}{\hookrightarrow}$ is a function from B into A.

The following proposition is true

(2) For every non-empty set A and for every non-empty subset B of A and for every element x of B holds $(\overset{B}{\hookrightarrow})(x) = x$.

In the sequel A denotes a set. Let us consider A. Let us assume that A is non-empty. The functor $[A]$ yielding an non-empty set is defined by:

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(Def.2) $[A] = A$.

We follow the rules: B, C will denote elements of $\operatorname{Fin} \mathrm{DP}(A)$, a, b, c, s, t_1, t_2 will denote elements of $\mathrm{DP}(A)$, and u, v, w will denote elements of the carrier of the lattice of normal forms over A. The following propositions are true:

(3) If $B = \emptyset$, then $\mu B = \emptyset$.

(4) For an arbitrary x such that $x \in B$ holds x is an element of $\mathrm{DP}(A)$.

Let us consider A, a. Then $\{a\}$ is an element of the normal forms over A.

Let us consider A, and let u be an element of the carrier of the lattice of normal forms over A.

The functor ${}^{@}u$ yields an element of the normal forms over A and is defined as follows:

(Def.3) ${}^{@}u = u$.

One can prove the following two propositions:

(5) $\sqcap_A({}^{@}u, {}^{@}v) =$ (the meet operation of the lattice of normal forms over A)(u, v).

(6) $\sqcup_A({}^{@}u, {}^{@}v) =$ (the join operation of the lattice of normal forms over A)(u, v).

In the sequel K, L will denote elements of the normal forms over A. One can prove the following propositions:

(7) $\mu(K \frown K) = K$.

(8) For every set X such that $X \subseteq K$ holds $X \in$ the normal forms over A.

(9) $\emptyset$ is an element of the normal forms over A.

(10) For every set X such that $X \subseteq u$ holds X is an element of the carrier of the lattice of normal forms over A.

Let us consider A. The functor $\{\square\}_A$ yields a function from $\mathrm{DP}(A)$ into the carrier of the lattice of normal forms over A and is defined by:

(Def.4) $\{\square\}_A(a) = \{a\}$.

The following propositions are true:

(11) If $c \in \{\square\}_A(a)$, then $c = a$.

(12) $a \in \{\square\}_A(a)$.

(13) $\{\square\}_A(a) = \mathrm{singleton}_{\mathrm{DP}(A)}(a)$.

(14) $\bigsqcup^{\mathrm{f}}_K(\{\square\}_A) = \mathrm{FinUnion}(K, \mathrm{singleton}_{\mathrm{DP}(A)})$.

(15) $u = \bigsqcup^{\mathrm{f}}_{({}^{@}u)}(\{\square\}_A)$.

In the sequel f will denote an element of $[\!:\operatorname{Fin} A, \operatorname{Fin} A:\!]^{\mathrm{DP}(A)}$ and g will denote an element of $[A]^{\mathrm{DP}(A)}$. Let A be a set. The functor $\square \setminus_A \square$ yielding a binary operation on $[\!:\operatorname{Fin} A, \operatorname{Fin} A:\!]$ is defined as follows:

(Def.5) for all elements a, b of $[\!:\operatorname{Fin} A, \operatorname{Fin} A:\!]$ holds $\square \setminus_A \square(a, b) = a \setminus b$.

We now define two new functors. Let us consider A, B. The functor $-B$ yielding an element of $\operatorname{Fin} \mathrm{DP}(A)$ is defined by:

(Def.6) $-B = \mathrm{DP}(A) \cap \{\langle\{g(t_1) : g(t_1) \in t_{\mathbf{12}} \wedge t_1 \in B\},$
$\{g(t_2) : g(t_2) \in t_{\mathbf{21}} \wedge t_2 \in B\}\rangle : s \in B \Rightarrow g(s) \in s_{\mathbf{1}} \cup s_{\mathbf{2}}\}$.

Let us consider C. The functor $B \rightarrowtail C$ yielding an element of $\mathrm{Fin}\,\mathrm{DP}(A)$ is defined by:

(Def.7) $B \rightarrowtail C = \mathrm{DP}(A) \cap \Big\{\mathrm{FinUnion}(B, \square \setminus_A \square^\circ(f, \overset{\mathrm{DP}(A)}{\hookrightarrow})) : f \circ B \subseteq C\Big\}$.

The following propositions are true:

(16) Suppose $c \in -B$. Then there exists g such that for every s such that $s \in B$ holds $g(s) \in s_{\mathbf{1}} \cup s_{\mathbf{2}}$ and $c = \langle\, \{g(t_1) : g(t_1) \in t_{\mathbf{12}} \wedge t_1 \in B\}\,, \{g(t_2) : g(t_2) \in t_{\mathbf{21}} \wedge t_2 \in B\}\,\rangle$.

(17) $\langle\emptyset, \emptyset\rangle$ is an element of $\mathrm{DP}(A)$.

(18) For every K such that $K = \emptyset$ holds $-K = \{\langle\emptyset, \emptyset\rangle\}$.

(19) For all K, L such that $K = \emptyset$ and $L = \emptyset$ holds $K \rightarrowtail L = \{\langle\emptyset, \emptyset\rangle\}$.

(20) For every element a of $\mathrm{DP}(\emptyset)$ holds $a = \langle\emptyset, \emptyset\rangle$.

(21) $\mathrm{DP}(\emptyset) = \{\langle\emptyset, \emptyset\rangle\}$.

(22) $\{\langle\emptyset, \emptyset\rangle\}$ is an element of the normal forms over A.

(23) If $c \in B \rightarrowtail C$, then there exists f such that $f \circ B \subseteq C$ and $c = \mathrm{FinUnion}(B, \square \setminus_A \square^\circ(f, \overset{\mathrm{DP}(A)}{\hookrightarrow}))$.

(24) If $K \frown \{a\} = \emptyset$, then there exists b such that $b \in -K$ and $b \subseteq a$.

(25) If for every b such that $b \in u$ holds $b \cup a \in \mathrm{DP}(A)$ and for every c such that $c \in u$ there exists b such that $b \in v$ and $b \subseteq c \cup a$, then there exists b such that $b \in ({}^{@}u) \rightarrowtail {}^{@}v$ and $b \subseteq a$.

(26) $K \frown -K = \emptyset$.

We now define four new functors. Let us consider A. The functor $\square^{\mathrm{c}}{}_A$ yielding a unary operation on the carrier of the lattice of normal forms over A is defined by:

(Def.8) $\square^{\mathrm{c}}{}_A(u) = \mu(-{}^{@}u)$.

The functor $\square \rightarrowtail_A \square$ yields a binary operation on the carrier of the lattice of normal forms over A and is defined by:

(Def.9) $(\square \rightarrowtail_A \square)(u, v) = \mu(({}^{@}u) \rightarrowtail {}^{@}v)$.

Let us consider u. The functor 2^u yielding an element of Fin (the carrier of the lattice of normal forms over A) is defined by:

(Def.10) $2^u = 2^u$.

The functor $\square \setminus_u \square$ yielding a unary operation on the carrier of the lattice of normal forms over A is defined as follows:

(Def.11) $(\square \setminus_u \square)(v) = u \setminus v$.

We now state several propositions:

(27) $(\square \setminus_u \square)(v) \sqsubseteq u$.

(28) $u \sqcap \square^{\mathrm{c}}{}_A(u) = \bot_{\text{the lattice of normal forms over } A}$.

(29) $u \sqcap (\square \rightarrowtail_A \square)(u, v) \sqsubseteq v$.

(30) If $({}^{@}u) \frown \{a\} = \emptyset$, then $\{\square\}_A(a) \sqsubseteq \square^{c}{}_A(u)$.

(31) If for every b such that $b \in u$ holds $b \cup a \in \mathrm{DP}(A)$ and $u \sqcap \{\square\}_A(a) \sqsubseteq w$, then $\{\square\}_A(a) \sqsubseteq (\square \rightarrowtail_A \square)(u, w)$.

(32) The lattice of normal forms over A is an implicative lattice.

(33) $u \Rightarrow v = \bigsqcup^{\mathrm{f}}_{2^u}($ (the meet operation of the lattice of normal forms over $A)^{\circ}(\square^{c}{}_A, (\square \rightarrowtail_A \square)^{\circ}(\square \setminus_u \square, v)))$.

(34) $\top_{\text{The lattice of normal forms over } A} = \{\langle \emptyset, \emptyset \rangle\}$.

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König's Lemma [1]

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Summary. A contiuation of [5]. The notions of finite-order trees, succesors of an element of a tree, and chains, levels and branches of a tree are introduced. Those notions are used to formalize König's Lemma which claims that there is a infinite branch of a finite-order tree if the tree has arbitrary long finite chains. Besides, the concept of decorated trees is introduced and some concepts dealing with trees are applied to decorated trees.

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The articles [12], [7], [10], [4], [6], [9], [2], [1], [3], [8], [11], [13], and [5] provide the notation and terminology for this paper. For simplicity we adopt the following rules: x, y are arbitrary, W, W_1, W_2 denote trees, w denotes an element of W, X denotes a set, f, f_1, f_2 denote functions, D, D' denote non-empty sets, k, k_1, k_2, m, n denote natural numbers, v, v_1, v_2 denote finite sequences, and p, q, r denote finite sequences of elements of $\mathbb{N}$. The following propositions are true:

(1) For all v_1, v_2, v such that $v_1 \preceq v$ and $v_2 \preceq v$ holds v_1 and v_2 are comparable.

(2) For all v_1, v_2, v such that $v_1 \prec v$ and $v_2 \preceq v$ holds v_1 and v_2 are comparable and v_2 and v_1 are comparable.

(4)[2] If $\operatorname{len} v_1 = k + 1$, then there exist v_2, x such that $v_1 = v_2 \frown \langle x \rangle$ and $\operatorname{len} v_2 = k$.

(5) $(v_1 \frown v_2) \restriction \operatorname{Seg} \operatorname{len} v_1 = v_1$.

(6) $\operatorname{Seg}_{\preceq}(v \frown \langle x \rangle) = \operatorname{Seg}_{\preceq}(v) \cup \{v\}$.

The scheme *TreeStruct_Ind* concerns a tree $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

for every element t of $\mathcal{A}$ holds $\mathcal{P}[t]$

[1]Partially supported by RPBP.III-24.C1

[2]The proposition (3) was either repeated or obvious.

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provided the following requirements are met:

- $\mathcal{P}[\varepsilon]$,
- for every element t of $\mathcal{A}$ and for every n such that $\mathcal{P}[t]$ and $t ^\frown \langle n \rangle \in \mathcal{A}$ holds $\mathcal{P}[t ^\frown \langle n \rangle]$.

We now state the proposition

(7) If for every p holds $p \in W_1$ if and only if $p \in W_2$, then $W_1 = W_2$.

Let us consider W_1, W_2. Let us note that one can characterize the predicate $W_1 = W_2$ by the following (equivalent) condition:

(Def.1) for every p holds $p \in W_1$ if and only if $p \in W_2$.

One can prove the following propositions:

(8) If $p \in W$, then $W = W(p/(W \restriction p))$.

(9) If $p \in W$ and $q \in W$ and $p \npreceq q$, then $q \in W(p/W_1)$.

(10) If $p \in W$ and $q \in W$ and p and q are not comparable, then $W(p/W_1)(q/W_2) = W(q/W_2)(p/W_1)$.

A tree is finite-order if:

(Def.2) there exists n such that for every element t of it holds $t ^\frown \langle n \rangle \notin$ it.

We now define three new constructions. Let us consider W. A subset of W is said to be a chain of W if:

(Def.3) for all p, q such that $p \in$ it and $q \in$ it holds p and q are comparable.

A subset of W is called a level of W if:

(Def.4) there exists n such that it $= \{w : \operatorname{len} w = n\}$.

Let us consider w. The functor $\operatorname{succ} w$ yielding a subset of W is defined by:

(Def.5) $\operatorname{succ} w = \{w ^\frown \langle n \rangle : w ^\frown \langle n \rangle \in W\}$.

One can prove the following propositions:

(11) For every level L of W holds L is an antichain of prefixes of W.

(12) $\operatorname{succ} w$ is an antichain of prefixes of W.

(13) For every antichain A of prefixes of W and for every chain C of W there exists w such that $A \cap C \subseteq \{w\}$.

Let us consider W, n. The functor n_W yielding a level of W is defined by:

(Def.6) $n_W = \{w : \operatorname{len} w = n\}$.

We now state several propositions:

(14) $w ^\frown \langle n \rangle \in \operatorname{succ} w$ if and only if $w ^\frown \langle n \rangle \in W$.

(15) If $w = \varepsilon$, then $1_W = \operatorname{succ} w$.

(16) $W = \bigcup\{n_W\}$.

(17) For every finite tree W holds $W = \bigcup\{n_W : n \leq \operatorname{height} W\}$.

(18) For every level L of W there exists n such that $L = n_W$.

Now we present three schemes. The scheme *AuxSch* concerns a tree $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

$\{w : \mathcal{P}[w]\}$, where w ranges over elements of $\mathcal{A}$, is a subset of $\mathcal{A}$

for all values of the parameters.

The scheme *FraenkelCard* concerns a non-empty set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ and states that:

$\overline{\overline{\{\mathcal{F}(w) : w \in \mathcal{B}\}}} \leq \overline{\overline{\mathcal{B}}}$, where w ranges over elements of $\mathcal{A}$

for all values of the parameters.

The scheme *FraenkelFinCard* concerns a non-empty set $\mathcal{A}$, a set $\mathcal{B}$, and a unary functor $\mathcal{F}$ and states that:

$\mathrm{card}\{\mathcal{F}(w) : w \in \mathcal{B}\} \leq \mathrm{card}\,\mathcal{B}$, where w ranges over elements of $\mathcal{A}$

provided the parameters meet the following requirement:

- $\mathcal{B}$ is finite.

The following four propositions are true:

(19) If W is finite-order, then there exists n such that for every w holds $\mathrm{succ}\,w$ is finite and $\mathrm{card}\,\mathrm{succ}\,w \leq n$.

(20) If W is finite-order, then $\mathrm{succ}\,w$ is finite.

(21) $\emptyset$ is a chain of W.

(22) $\{\varepsilon\}$ is a chain of W.

Let us consider W. A chain of W is said to be a branch of W if:

(Def.7) for every p such that $p \in$ it holds $\mathrm{Seg}_{\preceq}(p) \subseteq$ it and for no p holds $p \in W$ and for every q such that $q \in$ it holds $q \prec p$.

Let us consider W. We see that the branch of W is an non-empty chain of W.

In the sequel C will be a chain of W and B will be a branch of W. The following propositions are true:

(23) If $v_1 \in C$ and $v_2 \in C$, then $v_1 \in \mathrm{Seg}_{\preceq}(v_2)$ or $v_2 \preceq v_1$.

(24) If $v_1 \in C$ and $v_2 \in C$ and $v \preceq v_2$, then $v_1 \in \mathrm{Seg}_{\preceq}(v)$ or $v \preceq v_1$.

(25) If C is finite and $\mathrm{card}\,C > n$, then there exists p such that $p \in C$ and $\mathrm{len}\,p \geq n$.

(26) For every C holds $\{w : \bigvee_p [p \in C \wedge w \preceq p]\}$ is a chain of W.

(27) If $p \preceq q$ and $q \in B$, then $p \in B$.

(28) $\varepsilon \in B$.

(29) If $p \in C$ and $q \in C$ and $\mathrm{len}\,p \leq \mathrm{len}\,q$, then $p \preceq q$.

(30) There exists B such that $C \subseteq B$.

Now we present two schemes. The scheme *FuncExOfMinNat* concerns a set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists f such that $\mathrm{dom}\,f = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ there exists n such that $f(x) = n$ and $\mathcal{P}[x, n]$ and for every m such that $\mathcal{P}[x, m]$ holds $n \leq m$

provided the following condition is met:

- for every x such that $x \in \mathcal{A}$ there exists n such that $\mathcal{P}[x, n]$.

The scheme *InfiniteChain* concerns a set $\mathcal{A}$, a constant $\mathcal{B}$, a unary predicate $\mathcal{P}$, and a binary predicate $\mathcal{Q}$, and states that:

there exists f such that $\mathrm{dom}\,f = \mathbb{N}$ and $\mathrm{rng}\,f \subseteq \mathcal{A}$ and $f(0) = \mathcal{B}$ and for every k holds $\mathcal{Q}[f(k), f(k+1)]$ and $\mathcal{P}[f(k)]$

provided the parameters meet the following conditions:

- $\mathcal{B} \in \mathcal{A}$ and $\mathcal{P}[\mathcal{B}]$,
- for every x such that $x \in \mathcal{A}$ and $\mathcal{P}[x]$ there exists y such that $y \in \mathcal{A}$ and $\mathcal{Q}[x, y]$ and $\mathcal{P}[y]$.

The following two propositions are true:

(31) For every tree T such that for every n there exists a chain C of T such that C is finite and $\operatorname{card} C = n$ and for every element t of T holds $\operatorname{succ} t$ is finite there exists a chain B of T such that B is not finite.

(32) For every finite-order tree T such that for every n there exists a chain C of T such that C is finite and $\operatorname{card} C = n$ there exists a chain B of T such that B is not finite.

A function is said to be a decorated tree if:

(Def.8) dom it is a tree.

In the sequel T, T_1, T_2 are decorated trees. Let us consider T. Then $\operatorname{dom} T$ is a tree.

Let us consider D. A decorated tree is said to be a tree decorated by D if:

(Def.9) $\operatorname{rng} \mathrm{it} \subseteq D$.

Let D be a non-empty set, and let T be a tree decorated by D, and let t be an element of $\operatorname{dom} T$. Then $T(t)$ is an element of D.

One can prove the following proposition

(33) If $\operatorname{dom} T_1 = \operatorname{dom} T_2$ and for every p such that $p \in \operatorname{dom} T_1$ holds $T_1(p) = T_2(p)$, then $T_1 = T_2$.

Now we present two schemes. The scheme *DTreeEx* concerns a tree $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists T such that $\operatorname{dom} T = \mathcal{A}$ and for every p such that $p \in \mathcal{A}$ holds $\mathcal{P}[p, T(p)]$

provided the following condition is satisfied:

- for every p such that $p \in \mathcal{A}$ there exists x such that $\mathcal{P}[p, x]$.

The scheme *DTreeLambda* deals with a tree $\mathcal{A}$ and a unary functor $\mathcal{F}$ and states that:

there exists T such that $\operatorname{dom} T = \mathcal{A}$ and for every p such that $p \in \mathcal{A}$ holds $T(p) = \mathcal{F}(p)$

for all values of the parameters.

We now define two new functors. Let us consider T. The functor $\operatorname{Leaves} T$ yielding a set is defined by:

(Def.10) $\operatorname{Leaves} T = T^\circ \operatorname{Leaves} \operatorname{dom} T$.

Let us consider p. The functor $T \upharpoonright p$ yielding a decorated tree is defined by:

(Def.11) $\operatorname{dom}(T \upharpoonright p) = \operatorname{dom} T \upharpoonright p$ and for every q such that $q \in \operatorname{dom} T \upharpoonright p$ holds $(T \upharpoonright p)(q) = T(p \frown q)$.

The following proposition is true

(34) If $p \in \operatorname{dom} T$, then $\operatorname{rng}(T \upharpoonright p) \subseteq \operatorname{rng} T$.

Let us consider D, and let T be a tree decorated by D. Then $\mathrm{Leaves}\,T$ is a subset of D. Let p be an element of $\mathrm{dom}\,T$. Then $T \restriction p$ is a tree decorated by D.

Let us consider T, p, T_1. Let us assume that $p \in \mathrm{dom}\,T$. The functor $T(p/T_1)$ yielding a decorated tree is defined by the conditions (Def.12).

(Def.12) (i) $\mathrm{dom}(T(p/T_1)) = (\mathrm{dom}\,T)(p/\mathrm{dom}\,T_1)$,

(ii) for every q such that $q \in (\mathrm{dom}\,T)(p/\mathrm{dom}\,T_1)$ holds $p \not\preceq q$ and $T(p/T_1)(q) = T(q)$ or there exists r such that $r \in \mathrm{dom}\,T_1$ and $q = p \frown r$ and $T(p/T_1)(q) = T_1(r)$.

Let us consider W, x. Then $W \longmapsto x$ is a decorated tree.

Let D be a non-empty set, and let us consider W, and let d be an element of D. Then $W \longmapsto d$ is a tree decorated by D.

Next we state four propositions:

(35) If for every x such that $x \in D$ holds x is a tree, then $\bigcup D$ is a tree.

(36) If for every x such that $x \in X$ holds x is a function and for all f_1, f_2 such that $f_1 \in X$ and $f_2 \in X$ holds $\mathrm{graph}\,f_1 \subseteq \mathrm{graph}\,f_2$ or $\mathrm{graph}\,f_2 \subseteq \mathrm{graph}\,f_1$, then $\bigcup X$ is a function.

(37) If for every x such that $x \in D$ holds x is a decorated tree and for all T_1, T_2 such that $T_1 \in D$ and $T_2 \in D$ holds $\mathrm{graph}\,T_1 \subseteq \mathrm{graph}\,T_2$ or $\mathrm{graph}\,T_2 \subseteq \mathrm{graph}\,T_1$, then $\bigcup D$ is a decorated tree.

(38) If for every x such that $x \in D'$ holds x is a tree decorated by D and for all T_1, T_2 such that $T_1 \in D'$ and $T_2 \in D'$ holds $\mathrm{graph}\,T_1 \subseteq \mathrm{graph}\,T_2$ or $\mathrm{graph}\,T_2 \subseteq \mathrm{graph}\,T_1$, then $\bigcup D'$ is a tree decorated by D.

Now we present two schemes. The scheme *DTreeStructEx* deals with a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a function $\mathcal{C}$ from $[\!:\mathcal{A}, \mathbb{N}:\!]$ into $\mathcal{A}$ and states that:

there exists a tree T decorated by $\mathcal{A}$ such that $T(\varepsilon) = \mathcal{B}$ and for every element t of $\mathrm{dom}\,T$ holds $\mathrm{succ}\,t = \{t \frown \langle k \rangle : k \in \mathcal{F}(T(t))\}$ and for all n, x such that $x = T(t)$ and $n \in \mathcal{F}(x)$ holds $T(t \frown \langle n \rangle) = \mathcal{C}(\langle x, n \rangle)$

provided the following condition is satisfied:

- for every element d of $\mathcal{A}$ and for all k_1, k_2 such that $k_1 \leq k_2$ and $k_2 \in \mathcal{F}(d)$ holds $k_1 \in \mathcal{F}(d)$.

The scheme *DTreeStructFinEx* deals with a non-empty set $\mathcal{A}$, an element $\mathcal{B}$ of $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a natural number, and a function $\mathcal{C}$ from $[\!:\mathcal{A}, \mathbb{N}:\!]$ into $\mathcal{A}$ and states that:

there exists a tree T decorated by $\mathcal{A}$ such that $T(\varepsilon) = \mathcal{B}$ and for every element t of $\mathrm{dom}\,T$ holds $\mathrm{succ}\,t = \{t \frown \langle k \rangle : k < \mathcal{F}(T(t))\}$ and for all n, x such that $x = T(t)$ and $n < \mathcal{F}(x)$ holds $T(t \frown \langle n \rangle) = \mathcal{C}(\langle x, n \rangle)$

for all values of the parameters.

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Monotonic and Continuous Real Function

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Summary. A continuation of [16] and [13]. We prove a few theorems about real functions monotonic and continuous on interval, on halfline and on the set of real numbers and continuity of the inverse function. At the begining of the paper we show some facts about topological properties of the set of real numbers, halflines and intervals which rather belong to [17]

MML Identifier: `FCONT_3`.

The notation and terminology used in this paper are introduced in the following articles: [18], [5], [1], [2], [3], [20], [12], [6], [8], [15], [14], [4], [19], [9], [10], [17], [11], [16], and [7]. For simplicity we follow the rules: X will denote a set, x_0, r, r_1, g, p will denote real numbers, n will denote a natural number, a will denote a sequence of real numbers, and f will denote a partial function from $\mathbb{R}$ to $\mathbb{R}$. Next we state several propositions:

(1) $\Omega_{\mathbb{R}}$ is closed.

(2) $\emptyset_{\mathbb{R}}$ is open.

(3) $\emptyset_{\mathbb{R}}$ is closed.

(4) $\Omega_{\mathbb{R}}$ is open.

(5) $[r, +\infty[$ is closed.

(6) $]-\infty, r]$ is closed.

(7) $]r, +\infty[$ is open.

(8) $]-\infty, r[$ is open.

Let us consider r. Then $]r, +\infty[$ is a real open subset. Then $\mathrm{HL}(r)$ is a real open subset.

Let us consider p, g. Then $]p, g[$ is a real open subset.

Next we state a number of propositions:

(9) $0 < r$ and $g \in]x_0 - r, x_0 + r[$ if and only if there exists r_1 such that $g = x_0 + r_1$ and $|r_1| < r$.

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(10) $0 < r$ and $g \in]x_0 - r, x_0 + r[$ if and only if $g - x_0 \in]-r, r[$.

(11) $]-\infty, p] = \{p\} \cup]-\infty, p[$.

(12) $[p, +\infty[= \{p\} \cup]p, +\infty[$.

(13) If for every n holds $a(n) = x_0 - \frac{p}{n+1}$, then a is convergent and $\lim a = x_0$.

(14) If for every n holds $a(n) = x_0 + \frac{p}{n+1}$, then a is convergent and $\lim a = x_0$.

(15) If f is continuous in x_0 and $f(x_0) \neq r$ and there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$, then there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and for every g such that $g \in N$ holds $f(g) \neq r$.

(16) If f is increasing on X or f is decreasing on X, then $f \restriction X$ is one-to-one.

(17) If f is increasing on X, then $(f \restriction X)^{-1}$ is increasing on $f \circ X$.

(18) If f is decreasing on X, then $(f \restriction X)^{-1}$ is decreasing on $f \circ X$.

(19) If $X \subseteq \operatorname{dom} f$ and f is monotone on X and there exists p such that $f \circ X =]-\infty, p[$, then f is continuous on X.

(20) If $X \subseteq \operatorname{dom} f$ and f is monotone on X and there exists p such that $f \circ X =]p, +\infty[$, then f is continuous on X.

(21) If $X \subseteq \operatorname{dom} f$ and f is monotone on X and there exists p such that $f \circ X =]-\infty, p]$, then f is continuous on X.

(22) If $X \subseteq \operatorname{dom} f$ and f is monotone on X and there exists p such that $f \circ X = [p, +\infty[$, then f is continuous on X.

(23) If $X \subseteq \operatorname{dom} f$ and f is monotone on X and there exist p, g such that $f \circ X =]p, g[$, then f is continuous on X.

(24) If $X \subseteq \operatorname{dom} f$ and f is monotone on X and $f \circ X = \mathbb{R}$, then f is continuous on X.

(25) If f is increasing on $]p, g[$ or f is decreasing on $]p, g[$ but $]p, g[\subseteq \operatorname{dom} f$, then $(f \restriction]p, g[)^{-1}$ is continuous on $f \circ]p, g[$.

(26) If f is increasing on $]-\infty, p[$ or f is decreasing on $]-\infty, p[$ but $]-\infty, p[\subseteq \operatorname{dom} f$, then $(f \restriction]-\infty, p[)^{-1}$ is continuous on $f \circ]-\infty, p[$.

(27) If f is increasing on $]p, +\infty[$ or f is decreasing on $]p, +\infty[$ but $]p, +\infty[\subseteq \operatorname{dom} f$, then $(f \restriction]p, +\infty[)^{-1}$ is continuous on $f \circ]p, +\infty[$.

(28) If f is increasing on $]-\infty, p]$ or f is decreasing on $]-\infty, p]$ but $]-\infty, p] \subseteq \operatorname{dom} f$, then $(f \restriction]-\infty, p])^{-1}$ is continuous on $f \circ]-\infty, p]$.

(29) If f is increasing on $[p, +\infty[$ or f is decreasing on $[p, +\infty[$ but $[p, +\infty[\subseteq \operatorname{dom} f$, then $(f \restriction [p, +\infty[)^{-1}$ is continuous on $f \circ [p, +\infty[$.

(30) If f is increasing on $\Omega_{\mathbb{R}}$ or f is decreasing on $\Omega_{\mathbb{R}}$ but f is total, then f^{-1} is continuous on $\operatorname{rng} f$.

(31) If f is continuous on $]p, g[$ but f is increasing on $]p, g[$ or f is decreasing on $]p, g[$, then $\operatorname{rng}(f \restriction]p, g[)$ is open.

(32) If f is continuous on $]-\infty, p[$ but f is increasing on $]-\infty, p[$ or f is decreasing on $]-\infty, p[$, then $\operatorname{rng}(f \restriction]-\infty, p[)$ is open.

(33) If f is continuous on $]p, +\infty[$ but f is increasing on $]p, +\infty[$ or f is decreasing on $]p, +\infty[$, then $\operatorname{rng}(f \restriction]p, +\infty[)$ is open.

(34) If f is continuous on $\Omega_{\mathbb{R}}$ but f is increasing on $\Omega_{\mathbb{R}}$ or f is decreasing on $\Omega_{\mathbb{R}}$, then $\operatorname{rng} f$ is open.

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Real Function Differentiability - Part II

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Summary. A continuation of [18]. We prove an equivalent definition of the derivative of the real function at the point and theorems about derivative of composite functions, inverse function and derivative of quotient of two functions. At the begining of the paper a few facts which rather belong to [8], [10], [7] are proved.

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The terminology and notation used in this paper have been introduced in the following papers: [20], [5], [1], [2], [3], [22], [14], [8], [10], [16], [15], [4], [21], [11], [12], [19], [13], [17], [18], [9], and [6]. For simplicity we adopt the following convention: x_0, r, r_1, r_2, g, p will be real numbers, n, m will be natural numbers, a, b, d will be sequences of real numbers, h, h_1, h_2 will be real sequences convergent to 0, c will be a constant real sequence, A will be a real open subset, and f, f_1, f_2 will be partial functions from $\mathbb{R}$ to $\mathbb{R}$. Let us consider h. Then $-h$ is a real sequence convergent to 0.

The following propositions are true:

(1) If a is convergent and b is convergent and $\lim a = \lim b$ and for every n holds $d(2 \cdot n) = a(n)$ and $d(2 \cdot n + 1) = b(n)$, then d is convergent and $\lim d = \lim a$.

(2) If for every n holds $a(n) = 2 \cdot n$, then a is an increasing sequence of naturals.

(3) If for every n holds $a(n) = 2 \cdot n + 1$, then a is an increasing sequence of naturals.

(4) If $\operatorname{rng} c = \{x_0\}$, then c is convergent and $\lim c = x_0$ and $h + c$ is convergent and $\lim(h + c) = x_0$.

(5) If $\operatorname{rng} a = \{r\}$ and $\operatorname{rng} b = \{r\}$, then $a = b$.

(6) If a is a subsequence of h, then a is a real sequence convergent to 0.

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(7) Suppose for all h, c such that $\operatorname{rng} c = \{g\}$ and $\operatorname{rng}(h+c) \subseteq \operatorname{dom} f$ and $\{g\} \subseteq \operatorname{dom} f$ holds $h^{-1}(f \cdot (h+c) - f \cdot c)$ is convergent. Given h_1, h_2, c. Suppose $\operatorname{rng} c = \{g\}$ and $\operatorname{rng}(h_1+c) \subseteq \operatorname{dom} f$ and $\operatorname{rng}(h_2+c) \subseteq \operatorname{dom} f$ and $\{g\} \subseteq \operatorname{dom} f$. Then $\lim({h_1}^{-1}(f \cdot (h_1+c) - f \cdot c)) = \lim({h_2}^{-1}(f \cdot (h_2 + c) - f \cdot c))$.

(8) If there exists a neighbourhood N of r such that $N \subseteq \operatorname{dom} f$, then there exist h, c such that $\operatorname{rng} c = \{r\}$ and $\operatorname{rng}(h+c) \subseteq \operatorname{dom} f$ and $\{r\} \subseteq \operatorname{dom} f$.

(9) If $\operatorname{rng} a \subseteq \operatorname{dom}(f_2 \cdot f_1)$, then $\operatorname{rng} a \subseteq \operatorname{dom} f_1$ and $\operatorname{rng}(f_1 \cdot a) \subseteq \operatorname{dom} f_2$.

The scheme *ExInc_Seq_of_Nat* concerns a sequence of real numbers $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

there exists an increasing sequence q of naturals such that for every n holds $\mathcal{P}[(\mathcal{A} \cdot q)(n)]$ and for every n such that for every r such that $r = \mathcal{A}(n)$ holds $\mathcal{P}[r]$ there exists m such that $n = q(m)$

provided the following requirement is met:

- for every n there exists m such that $n \leq m$ and $\mathcal{P}[\mathcal{A}(m)]$.

One can prove the following propositions:

(10) If $f(x_0) \neq r$ and f is differentiable in x_0, then there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and for every g such that $g \in N$ holds $f(g) \neq r$.

(11) f is differentiable in x_0 if and only if there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$ and for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h+c) \subseteq \operatorname{dom} f$ holds $h^{-1}(f \cdot (h+c) - f \cdot c)$ is convergent.

(12) f is differentiable in x_0 and $f'(x_0) = g$ if and only if the following conditions are satisfied:

(i) there exists a neighbourhood N of x_0 such that $N \subseteq \operatorname{dom} f$,

(ii) for all h, c such that $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h+c) \subseteq \operatorname{dom} f$ holds $h^{-1}(f \cdot (h+c) - f \cdot c)$ is convergent and $\lim(h^{-1}(f \cdot (h+c) - f \cdot c)) = g$.

(13) If f_1 is differentiable in x_0 and f_2 is differentiable in $f_1(x_0)$, then $f_2 \cdot f_1$ is differentiable in x_0 and $(f_2 \cdot f_1)'(x_0) = {f_2}'(f_1(x_0)) \cdot {f_1}'(x_0)$.

(14) If $f_2(x_0) \neq 0$ and f_1 is differentiable in x_0 and f_2 is differentiable in x_0, then $\frac{f_1}{f_2}$ is differentiable in x_0 and $(\frac{f_1}{f_2})'(x_0) = \frac{{f_1}'(x_0) \cdot f_2(x_0) - {f_2}'(x_0) \cdot f_1(x_0)}{f_2(x_0)^{\mathbf{2}}}$.

(15) If $f(x_0) \neq 0$ and f is differentiable in x_0, then $\frac{1}{f}$ is differentiable in x_0 and $(\frac{1}{f})'(x_0) = -\frac{f'(x_0)}{f(x_0)^{\mathbf{2}}}$.

(16) If f is differentiable on A, then $f \restriction A$ is differentiable on A and $f'_{\restriction A} = (f \restriction A)'_{\restriction A}$.

(17) If f_1 is differentiable on A and f_2 is differentiable on A, then $f_1 + f_2$ is differentiable on A and $(f_1 + f_2)'_{\restriction A} = {f_1}'_{\restriction A} + {f_2}'_{\restriction A}$.

(18) If f_1 is differentiable on A and f_2 is differentiable on A, then $f_1 - f_2$ is differentiable on A and $(f_1 - f_2)'_{\restriction A} = {f_1}'_{\restriction A} - {f_2}'_{\restriction A}$.

(19) If f is differentiable on A, then rf is differentiable on A and $(rf)'_{\restriction A} = rf'_{\restriction A}$.

(20) If f_1 is differentiable on A and f_2 is differentiable on A, then $f_1 f_2$ is differentiable on A and $(f_1 f_2)'_{\restriction A} = {f_1}'_{\restriction A} f_2 + f_1 {f_2}'_{\restriction A}$.

(21) If f_1 is differentiable on A and f_2 is differentiable on A and for every x_0 such that $x_0 \in A$ holds $f_2(x_0) \neq 0$, then $\frac{f_1}{f_2}$ is differentiable on A and $(\frac{f_1}{f_2})'_{\restriction A} = \frac{{f_1}'_{\restriction A} f_2 - {f_2}'_{\restriction A} f_1}{f_2 f_2}$.

(22) If f is differentiable on A and for every x_0 such that $x_0 \in A$ holds $f(x_0) \neq 0$, then $\frac{1}{f}$ is differentiable on A and $(\frac{1}{f})'_{\restriction A} = -\frac{f'_{\restriction A}}{f f}$.

(23) If f_1 is differentiable on A and $f_1 \circ A$ is a real open subset and f_2 is differentiable on $f_1 \circ A$, then $f_2 \cdot f_1$ is differentiable on A and $(f_2 \cdot f_1)'_{\restriction A} = ({f_2}'_{\restriction f_1 \circ A} \cdot f_1) {f_1}'_{\restriction A}$.

(24) If $A \subseteq \operatorname{dom} f$ and for all r, p such that $r \in A$ and $p \in A$ holds $|f(r) - f(p)| \leq (r - p)^{\mathbf{2}}$, then f is differentiable on A and for every x_0 such that $x_0 \in A$ holds $f'(x_0) = 0$.

(25) Suppose for all r_1, r_2 such that $r_1 \in]p, g[$ and $r_2 \in]p, g[$ holds $|f(r_1) - f(r_2)| \leq (r_1 - r_2)^{\mathbf{2}}$ and $p < g$ and $]p, g[\subseteq \operatorname{dom} f$. Then f is differentiable on $]p, g[$ and f is a constant on $]p, g[$.

(26) If $]-\infty, r[\subseteq \operatorname{dom} f$ and for all r_1, r_2 such that $r_1 \in]-\infty, r[$ and $r_2 \in]-\infty, r[$ holds $|f(r_1) - f(r_2)| \leq (r_1 - r_2)^{\mathbf{2}}$, then f is differentiable on $]-\infty, r[$ and f is a constant on $]-\infty, r[$.

(27) If $]r, +\infty[\subseteq \operatorname{dom} f$ and for all r_1, r_2 such that $r_1 \in]r, +\infty[$ and $r_2 \in]r, +\infty[$ holds $|f(r_1) - f(r_2)| \leq (r_1 - r_2)^{\mathbf{2}}$, then f is differentiable on $]r, +\infty[$ and f is a constant on $]r, +\infty[$.

(28) If f is total and for all r_1, r_2 holds $|f(r_1) - f(r_2)| \leq (r_1 - r_2)^{\mathbf{2}}$, then f is differentiable on $\Omega_{\mathbb{R}}$ and f is a constant on $\Omega_{\mathbb{R}}$.

(29) If f is differentiable on $]-\infty, r[$ and for every x_0 such that $x_0 \in]-\infty, r[$ holds $0 < f'(x_0)$, then f is increasing on $]-\infty, r[$ and $f \restriction]-\infty, r[$ is one-to-one.

(30) If f is differentiable on $]-\infty, r[$ and for every x_0 such that $x_0 \in]-\infty, r[$ holds $f'(x_0) < 0$, then f is decreasing on $]-\infty, r[$ and $f \restriction]-\infty, r[$ is one-to-one.

(31) If f is differentiable on $]-\infty, r[$ and for every x_0 such that $x_0 \in]-\infty, r[$ holds $0 \leq f'(x_0)$, then f is non-decreasing on $]-\infty, r[$.

(32) If f is differentiable on $]-\infty, r[$ and for every x_0 such that $x_0 \in]-\infty, r[$ holds $f'(x_0) \leq 0$, then f is non-increasing on $]-\infty, r[$.

(33) If f is differentiable on $]r, +\infty[$ and for every x_0 such that $x_0 \in]r, +\infty[$ holds $0 < f'(x_0)$, then f is increasing on $]r, +\infty[$ and $f \restriction]r, +\infty[$ is one-to-one.

(34) If f is differentiable on $]r, +\infty[$ and for every x_0 such that $x_0 \in]r, +\infty[$ holds $f'(x_0) < 0$, then f is decreasing on $]r, +\infty[$ and $f \restriction]r, +\infty[$ is one-to-one.

(35) If f is differentiable on $]r, +\infty[$ and for every x_0 such that $x_0 \in]r, +\infty[$

holds $0 \leq f'(x_0)$, then f is non-decreasing on $]r, +\infty[$.

(36) If f is differentiable on $]r, +\infty[$ and for every x_0 such that $x_0 \in]r, +\infty[$ holds $f'(x_0) \leq 0$, then f is non-increasing on $]r, +\infty[$.

(37) If f is differentiable on $\Omega_{\mathbb{R}}$ and for every x_0 holds $0 < f'(x_0)$, then f is increasing on $\Omega_{\mathbb{R}}$ and f is one-to-one.

(38) If f is differentiable on $\Omega_{\mathbb{R}}$ and for every x_0 holds $f'(x_0) < 0$, then f is decreasing on $\Omega_{\mathbb{R}}$ and f is one-to-one.

(39) If f is differentiable on $\Omega_{\mathbb{R}}$ and for every x_0 holds $0 \leq f'(x_0)$, then f is non-decreasing on $\Omega_{\mathbb{R}}$.

(40) If f is differentiable on $\Omega_{\mathbb{R}}$ and for every x_0 holds $f'(x_0) \leq 0$, then f is non-increasing on $\Omega_{\mathbb{R}}$.

One can prove the following propositions:

(41) If f is differentiable on $]p, g[$ but for every x_0 such that $x_0 \in]p, g[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in]p, g[$ holds $f'(x_0) < 0$, then $\operatorname{rng}(f \restriction]p, g[)$ is open.

(42) If f is differentiable on $]-\infty, p[$ but for every x_0 such that $x_0 \in]-\infty, p[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in]-\infty, p[$ holds $f'(x_0) < 0$, then $\operatorname{rng}(f \restriction]-\infty, p[)$ is open.

(43) If f is differentiable on $]p, +\infty[$ but for every x_0 such that $x_0 \in]p, +\infty[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in]p, +\infty[$ holds $f'(x_0) < 0$, then $\operatorname{rng}(f \restriction]p, +\infty[)$ is open.

(44) If f is differentiable on $\Omega_{\mathbb{R}}$ but for every x_0 holds $0 < f'(x_0)$ or for every x_0 holds $f'(x_0) < 0$, then $\operatorname{rng} f$ is open.

(45) Suppose f is differentiable on $\Omega_{\mathbb{R}}$ but for every x_0 holds $0 < f'(x_0)$ or for every x_0 holds $f'(x_0) < 0$. Then f is one-to-one and f^{-1} is differentiable on $\operatorname{dom}(f^{-1})$ and for every x_0 such that $x_0 \in \operatorname{dom}(f^{-1})$ holds $(f^{-1})'(x_0) = \frac{1}{f'(f^{-1}(x_0))}$.

(46) Suppose f is differentiable on $]-\infty, p[$ but for every x_0 such that $x_0 \in]-\infty, p[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in]-\infty, p[$ holds $f'(x_0) < 0$. Then $f \restriction]-\infty, p[$ is one-to-one and $(f \restriction]-\infty, p[)^{-1}$ is differentiable on $\operatorname{dom}((f \restriction]-\infty, p[)^{-1})$ and for every x_0 such that $x_0 \in \operatorname{dom}((f \restriction]-\infty, p[)^{-1})$ holds $((f \restriction]-\infty, p[)^{-1})'(x_0) = \frac{1}{f'((f \restriction]-\infty, p[)^{-1}(x_0))}$.

(47) Suppose f is differentiable on $]p, +\infty[$ but for every x_0 such that $x_0 \in]p, +\infty[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in]p, +\infty[$ holds $f'(x_0) < 0$. Then $f \restriction]p, +\infty[$ is one-to-one and $(f \restriction]p, +\infty[)^{-1}$ is differentiable on $\operatorname{dom}((f \restriction]p, +\infty[)^{-1})$ and for every x_0 such that $x_0 \in \operatorname{dom}((f \restriction]p, +\infty[)^{-1})$ holds $((f \restriction]p, +\infty[)^{-1})'(x_0) = \frac{1}{f'((f \restriction]p, +\infty[)^{-1}(x_0))}$.

(48) Suppose f is differentiable on $]p, g[$ but for every x_0 such that $x_0 \in]p, g[$ holds $0 < f'(x_0)$ or for every x_0 such that $x_0 \in]p, g[$ holds $f'(x_0) < 0$. Then

(i) $f \restriction]p, g[$ is one-to-one,

(ii) $(f \restriction]p, g[)^{-1}$ is differentiable on $\operatorname{dom}((f \restriction]p, g[)^{-1})$,

(iii) for every x_0 such that $x_0 \in \mathrm{dom}((f{\restriction}]p,g[)^{-1})$ holds $((f{\restriction}]p,g[)^{-1})'(x_0) = \frac{1}{f'((f{\restriction}]p,g[)^{-1}(x_0))}$.

(49) Suppose f is differentiable in x_0. Given h, c. Suppose $\mathrm{rng}\,c = \{x_0\}$ and $\mathrm{rng}(h+c) \subseteq \mathrm{dom}\,f$ and $\mathrm{rng}(-h+c) \subseteq \mathrm{dom}\,f$. Then $(2h)^{-1}(f\cdot(c+h) - f\cdot(c-h))$ is convergent and $\lim((2h)^{-1}(f\cdot(c+h)-f\cdot(c-h))) = f'(x_0)$.

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Preliminaries to the Lambek Calculus

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Summary. Some preliminary facts concerning completeness and decidability problems for the Lambek calculus [13] are proved as well as some theses and derived rules of the calculus itself.

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The articles [16], [7], [9], [10], [18], [6], [8], [12], [17], [15], [14], [5], [1], [11], [2], [3], and [4] provide the terminology and notation for this paper. We consider structures of the type algebra which are systems

⟨types, a left quotient, a right quotient, a inner product⟩,

where the types constitute a non-empty set and the left quotient, the right quotient, the inner product are a binary operation on the types.

Let s be a structure of the type algebra. A type of s is an element of the types of s.

We adopt the following rules: s will denote a structure of the type algebra, T, X, Y will denote finite sequences of elements of the types of s, and x, y, z will denote types of s. We now define three new functors. Let us consider s, x, y. The functor $x \setminus y$ yields a type of s and is defined by:

(Def.1) $x \setminus y =$ (the left quotient of s)(x, y).

The functor x/y yields a type of s and is defined as follows:

(Def.2) $x/y =$ (the right quotient of s)(x, y).

The functor $x \cdot y$ yields a type of s and is defined by:

(Def.3) $x \cdot y =$ (the inner product of s)(x, y).

Let T_1 be a tree, and let v be an element of T_1. The branch degree of v is defined by:

(Def.4) the branch degree of $v = \operatorname{card} \operatorname{succ} v$.

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Let us consider s. A preproof of s is a tree decorated by [: [: (the types of s)*, the types of s :], $\mathbb{N}$:].

In the sequel T_1 is a preproof of s. Let us consider s, T_1, and let v be an element of $\operatorname{dom} T_1$. We say that v is correct if and only if:

(Def.5) (i) the branch degree of $v = 0$ and there exists x such that $T_1(v)_{\mathbf{1}} = \langle\langle x\rangle, x\rangle$ if $T_1(v)_{\mathbf{2}} = 0$,

(ii) the branch degree of $v = 1$ and there exist T, x, y such that $T_1(v)_{\mathbf{1}} = \langle T, x/y\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle T \frown \langle y\rangle, x\rangle$ if $T_1(v)_{\mathbf{2}} = 1$,

(iii) the branch degree of $v = 1$ and there exist T, x, y such that $T_1(v)_{\mathbf{1}} = \langle T, y \setminus x\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle\langle y\rangle \frown T, x\rangle$ if $T_1(v)_{\mathbf{2}} = 2$,

(iv) the branch degree of $v = 2$ and there exist T, X, Y, x, y, z such that $T_1(v)_{\mathbf{1}} = \langle X \frown \langle x/y\rangle \frown T \frown Y, z\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle T, y\rangle$ and $T_1(v \frown \langle 1\rangle)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown Y, z\rangle$ if $T_1(v)_{\mathbf{2}} = 3$,

(v) the branch degree of $v = 2$ and there exist T, X, Y, x, y, z such that $T_1(v)_{\mathbf{1}} = \langle X \frown T \frown \langle y \setminus x\rangle \frown Y, z\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle T, y\rangle$ and $T_1(v \frown \langle 1\rangle)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown Y, z\rangle$ if $T_1(v)_{\mathbf{2}} = 4$,

(vi) the branch degree of $v = 1$ and there exist X, x, y, Y such that $T_1(v)_{\mathbf{1}} = \langle X \frown \langle x \cdot y\rangle \frown Y, z\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle X \frown \langle x\rangle \frown \langle y\rangle \frown Y, z\rangle$ if $T_1(v)_{\mathbf{2}} = 5$,

(vii) the branch degree of $v = 2$ and there exist X, Y, x, y such that $T_1(v)_{\mathbf{1}} = \langle X \frown Y, x \cdot y\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle X, x\rangle$ and $T_1(v \frown \langle 1\rangle)_{\mathbf{1}} = \langle Y, y\rangle$ if $T_1(v)_{\mathbf{2}} = 6$,

(viii) the branch degree of $v = 2$ and there exist T, X, Y, y, z such that $T_1(v)_{\mathbf{1}} = \langle X \frown T \frown Y, z\rangle$ and $T_1(v \frown \langle 0\rangle)_{\mathbf{1}} = \langle T, y\rangle$ and $T_1(v \frown \langle 1\rangle)_{\mathbf{1}} = \langle X \frown \langle y\rangle \frown Y, z\rangle$ if $T_1(v)_{\mathbf{2}} = 7$.

We now define three new attributes. Let us consider s. A type of s is left if:

(Def.6) there exist x, y such that it $= x \setminus y$.

A type of s is right if:

(Def.7) there exist x, y such that it $= x/y$.

A type of s is middle if:

(Def.8) there exist x, y such that it $= x \cdot y$.

Let us consider s. A type of s is primitive if:

(Def.9) neither it is left nor it is right nor it is middle.

Let us consider s, and let T_1 be a tree decorated by the types of s, and let us consider x. We say that T_1 represents x if and only if the conditions (Def.10) is satisfied.

(Def.10) (i) $\operatorname{dom} T_1$ is finite,

(ii) for every element v of $\operatorname{dom} T_1$ holds the branch degree of $v = 0$ or the branch degree of $v = 2$ but if the branch degree of $v = 0$, then $T_1(v)$ is primitive but if the branch degree of $v = 2$, then there exist y, z such that $T_1(v) = y/z$ or $T_1(v) = y \setminus z$ or $T_1(v) = y \cdot z$ but $T_1(v \frown \langle 0\rangle) = y$ and $T_1(v \frown \langle 1\rangle) = z$.

A structure of the type algebra is free if:

(Def.11) for no type x of it holds x is left right or x is left middle or x is right middle and for every type x of it there exists a tree T_1 decorated by the types of it such that for every tree T_2 decorated by the types of it holds T_2 represents x if and only if $T_1 = T_2$.

Let us consider s, x. Let us assume that s is free. The representation of x yields a tree decorated by the types of s and is defined by:

(Def.12) for every tree T_1 decorated by the types of s holds T_1 represents x if and only if the representation of $x = T_1$.

Let us consider s, and let f be a finite sequence of elements of the types of s, and let t be a type of s. Then $\langle f, t \rangle$ is an element of $[\!:$ (the types of $s)^*$, the types of $s:\!]$.

Let us consider s. A preproof of s is called a proof of s if:

(Def.13) dom it is a finite tree and for every element v of dom it holds v is correct.

In the sequel p is a proof of s and v is an element of dom p. The following propositions are true:

(1) If the branch degree of $v = 1$, then $v \frown \langle 0 \rangle \in \mathrm{dom}\, p$.

(2) If the branch degree of $v = 2$, then $v \frown \langle 0 \rangle \in \mathrm{dom}\, p$ and $v \frown \langle 1 \rangle \in \mathrm{dom}\, p$.

(3) If $p(v)_{\mathbf{2}} = 0$, then there exists x such that $p(v)_{\mathbf{1}} = \langle \langle x \rangle, x \rangle$.

(4) If $p(v)_{\mathbf{2}} = 1$, then there exists an element w of dom p and there exist T, x, y such that $w = v \frown \langle 0 \rangle$ and $p(v)_{\mathbf{1}} = \langle T, x/y \rangle$ and $p(w)_{\mathbf{1}} = \langle T \frown \langle y \rangle, x \rangle$.

(5) If $p(v)_{\mathbf{2}} = 2$, then there exists an element w of dom p and there exist T, x, y such that $w = v \frown \langle 0 \rangle$ and $p(v)_{\mathbf{1}} = \langle T, y \setminus x \rangle$ and $p(w)_{\mathbf{1}} = \langle \langle y \rangle \frown T, x \rangle$.

(6) Suppose $p(v)_{\mathbf{2}} = 3$. Then there exist elements w, u of dom p and there exist T, X, Y, x, y, z such that $w = v \frown \langle 0 \rangle$ and $u = v \frown \langle 1 \rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown \langle x/y \rangle \frown T \frown Y, z \rangle$ and $p(w)_{\mathbf{1}} = \langle T, y \rangle$ and $p(u)_{\mathbf{1}} = \langle X \frown \langle x \rangle \frown Y, z \rangle$.

(7) Suppose $p(v)_{\mathbf{2}} = 4$. Then there exist elements w, u of dom p and there exist T, X, Y, x, y, z such that $w = v \frown \langle 0 \rangle$ and $u = v \frown \langle 1 \rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown T \frown \langle y \setminus x \rangle \frown Y, z \rangle$ and $p(w)_{\mathbf{1}} = \langle T, y \rangle$ and $p(u)_{\mathbf{1}} = \langle X \frown \langle x \rangle \frown Y, z \rangle$.

(8) Suppose $p(v)_{\mathbf{2}} = 5$. Then there exists an element w of dom p and there exist X, x, y, Y such that $w = v \frown \langle 0 \rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown \langle x \cdot y \rangle \frown Y, z \rangle$ and $p(w)_{\mathbf{1}} = \langle X \frown \langle x \rangle \frown \langle y \rangle \frown Y, z \rangle$.

(9) Suppose $p(v)_{\mathbf{2}} = 6$. Then there exist elements w, u of dom p and there exist X, Y, x, y such that $w = v \frown \langle 0 \rangle$ and $u = v \frown \langle 1 \rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown Y, x \cdot y \rangle$ and $p(w)_{\mathbf{1}} = \langle X, x \rangle$ and $p(u)_{\mathbf{1}} = \langle Y, y \rangle$.

(10) Suppose $p(v)_{\mathbf{2}} = 7$. Then there exist elements w, u of dom p and there exist T, X, Y, y, z such that $w = v \frown \langle 0 \rangle$ and $u = v \frown \langle 1 \rangle$ and $p(v)_{\mathbf{1}} = \langle X \frown T \frown Y, z \rangle$ and $p(w)_{\mathbf{1}} = \langle T, y \rangle$ and $p(u)_{\mathbf{1}} = \langle X \frown \langle y \rangle \frown Y, z \rangle$.

(11) (i) $p(v)_{\mathbf{2}} = 0$, or
(ii) $p(v)_{\mathbf{2}} = 1$, or
(iii) $p(v)_{\mathbf{2}} = 2$, or
(iv) $p(v)_{\mathbf{2}} = 3$, or
(v) $p(v)_{\mathbf{2}} = 4$, or

(vi) $p(v)_{\mathbf{2}} = 5$, or
(vii) $p(v)_{\mathbf{2}} = 6$, or
(viii) $p(v)_{\mathbf{2}} = 7$.

We now define two new constructions. Let us consider s. A preproof of s is cut-free if:

(Def.14) for every element v of dom it holds $\mathrm{it}(v)_{\mathbf{2}} \neq 7$.

The size w.r.t. s yielding a function from the types of s into $\mathbb{N}$ is defined by:

(Def.15) for every x holds
(the size w.r.t. s)(x) = card dom(the representation of x).

Let D be a non-empty set, and let T be a finite sequence of elements of D, and let f be a function from D into $\mathbb{N}$. Then $f \cdot T$ is a finite sequence of elements of $\mathbb{R}$.

Let D be a non-empty set, and let f be a function from D into $\mathbb{N}$, and let d be an element of D. Then $f(d)$ is a natural number.

Let us consider s, and let p be a proof of s. Let us assume that s is free. The cut degree of p yields a natural number and is defined by:

(Def.16) (i) there exist T, X, Y, y, z such that $p(\varepsilon)_{\mathbf{1}} = \langle X \frown T \frown Y, z\rangle$ and $p(\langle 0\rangle)_{\mathbf{1}} = \langle T, y\rangle$ and $p(\langle 1\rangle)_{\mathbf{1}} = \langle X \frown \langle y\rangle \frown Y, z\rangle$ and the cut degree of p = (the size w.r.t. s)(y) + (the size w.r.t. s)(z) + $\sum$((the size w.r.t. s) $\cdot$ $(X \frown T \frown Y)$) if $p(\varepsilon)_{\mathbf{2}} = 7$,
(ii) the cut degree of $p = 0$, otherwise.

We adopt the following convention: A denotes an non-empty set and a, a_1, a_2, b denote elements of A^*. Let us consider s, A. A function from the types of s into 2^{A^*} is said to be a model of s if it satisfies the condition (Def.17).

(Def.17) Given x, y. Then
(i) $\mathrm{it}(x \cdot y) = \{a \frown b : a \in \mathrm{it}(x) \wedge b \in \mathrm{it}(y)\}$,
(ii) $\mathrm{it}(x/y) = \{a_1 : \bigwedge_b [b \in \mathrm{it}(y) \Rightarrow a_1 \frown b \in \mathrm{it}(x)]\}$,
(iii) $\mathrm{it}(y \setminus x) = \{a_2 : \bigwedge_b [b \in \mathrm{it}(y) \Rightarrow b \frown a_2 \in \mathrm{it}(x)]\}$.

We consider type structures which are systems
⟨structures of the type algebra; a derivability⟩,
where the derivability is a non-empty relation between
(the types of the structure of the type algebra)*
and the types of the structure of the type algebra.

In the sequel s will denote a type structure and x will denote a type of s. Let us consider s, and let f be a finite sequence of elements of the types of s, and let us consider x. The predicate $f \longrightarrow x$ is defined by:

(Def.18) $\langle f, x\rangle \in$ the derivability of s.

A type structure is called a calculus of syntactic types if it satisfies the conditions (Def.19).

(Def.19) (i) For every type x of it holds $\langle x\rangle \longrightarrow x$,
(ii) for every finite sequence T of elements of the types of it and for all types x, y of it such that $T \frown \langle y\rangle \longrightarrow x$ holds $T \longrightarrow x/y$,

(iii) for every finite sequence T of elements of the types of it and for all types x, y of it such that $\langle y\rangle \frown T \longrightarrow x$ holds $T \longrightarrow y \setminus x$,

(iv) for all finite sequences T, X, Y of elements of the types of it and for all types x, y, z of it such that $T \longrightarrow y$ and $X \frown \langle x\rangle \frown Y \longrightarrow z$ holds $X \frown \langle x/y\rangle \frown T \frown Y \longrightarrow z$,

(v) for all finite sequences T, X, Y of elements of the types of it and for all types x, y, z of it such that $T \longrightarrow y$ and $X \frown \langle x\rangle \frown Y \longrightarrow z$ holds $X \frown T \frown \langle y \setminus x\rangle \frown Y \longrightarrow z$,

(vi) for all finite sequences X, Y of elements of the types of it and for all types x, y, z of it such that $X \frown \langle x\rangle \frown \langle y\rangle \frown Y \longrightarrow z$ holds $X \frown \langle x \cdot y\rangle \frown Y \longrightarrow z$,

(vii) for all finite sequences X, Y of elements of the types of it and for all types x, y of it such that $X \longrightarrow x$ and $Y \longrightarrow y$ holds $X \frown Y \longrightarrow x \cdot y$.

In the sequel s will be a calculus of syntactic types and x, y, z will be types of s. The following propositions are true:

(12) $\langle x/y\rangle \frown \langle y\rangle \longrightarrow x$ and $\langle y\rangle \frown \langle y \setminus x\rangle \longrightarrow x$.

(13) $\langle x\rangle \longrightarrow y/(x \setminus y)$ and $\langle x\rangle \longrightarrow y/x \setminus y$.

(14) $\langle x/y\rangle \longrightarrow x/z/(y/z)$.

(15) $\langle y \setminus x\rangle \longrightarrow z \setminus y \setminus (z \setminus x)$.

(16) If $\langle x\rangle \longrightarrow y$, then $\langle x/z\rangle \longrightarrow y/z$ and $\langle z \setminus x\rangle \longrightarrow z \setminus y$.

(17) If $\langle x\rangle \longrightarrow y$, then $\langle z/y\rangle \longrightarrow z/x$ and $\langle y \setminus z\rangle \longrightarrow x \setminus z$.

(18) $\langle y/(y/x \setminus y)\rangle \longrightarrow y/x$.

(19) If $\langle x\rangle \longrightarrow y$, then $\varepsilon_{(\text{the types of } s)} \longrightarrow y/x$ and $\varepsilon_{(\text{the types of } s)} \longrightarrow x \setminus y$.

(20) $\varepsilon_{(\text{the types of } s)} \longrightarrow x/x$ and $\varepsilon_{(\text{the types of } s)} \longrightarrow x \setminus x$.

(21) $\varepsilon_{(\text{the types of } s)} \longrightarrow y/(x \setminus y)/x$ and $\varepsilon_{(\text{the types of } s)} \longrightarrow x \setminus (y/x \setminus y)$.

(22) $\varepsilon_{(\text{the types of } s)} \longrightarrow x/z/(y/z)/(x/y)$ and $\varepsilon_{(\text{the types of } s)} \longrightarrow y \setminus x \setminus (z \setminus y \setminus (z \setminus x))$.

(23) If $\varepsilon_{(\text{the types of } s)} \longrightarrow x$, then $\varepsilon_{(\text{the types of } s)} \longrightarrow y/(y/x)$ and $\varepsilon_{(\text{the types of } s)} \longrightarrow x \setminus y \setminus y$.

(24) $\langle x/(y/y)\rangle \longrightarrow x$.

Let us consider s, x, y. The predicate $x \longleftrightarrow y$ is defined as follows:

(Def.20) $\langle x\rangle \longrightarrow y$ and $\langle y\rangle \longrightarrow x$.

Next we state several propositions:

(25) $x \longleftrightarrow x$.

(26) $x/y \longleftrightarrow x/(x/y \setminus x)$.

(27) $x/(z \cdot y) \longleftrightarrow x/y/z$.

(28) $\langle x \cdot (y/z)\rangle \longrightarrow (x \cdot y)/z$.

(29) $\langle x\rangle \longrightarrow (x \cdot y)/y$ and $\langle x\rangle \longrightarrow y \setminus y \cdot x$.

(30) $x \cdot y \cdot z \longleftrightarrow x \cdot (y \cdot z)$.

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Opposite Categories and Contravariant Functors

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Summary. The opposite category of a category, contravariant functors and duality functors are defined.

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The articles [6], [1], [2], [5], [4], and [3] provide the notation and terminology for this paper. In the sequel B, C, D will be categories. Let X be a set, and let C, D be non-empty sets, and let f be a function from X into C, and let g be a function from C into D. Then $g \cdot f$ is a function from X into D.

Let X, Y, Z be non-empty sets, and let f be a partial function from $[\!:X, Y:\!]$ to Z. Then $\curvearrowleft f$ is a partial function from $[\!:Y, X:\!]$ to Z.

The following proposition is true

(1) $\langle$The objects of C, the morphisms of C, the cod-map of C, the dom-map of $C, \curvearrowleft$(the composition of C), the id-map of $C\rangle$ is a category.

Let us consider C. The functor $C^{\rm op}$ yielding a category is defined as follows:

(Def.1) $C^{\rm op} = \langle$ the objects of C, the morphisms of C, the cod-map of C, the dom-map of $C, \curvearrowleft$(the composition of C), the id-map of $C\rangle$.

One can prove the following proposition

(2) $(C^{\rm op})^{\rm op} = C$.

Let us consider C, and let c be an object of C. The functor $c^{\rm op}$ yields an object of $C^{\rm op}$ and is defined by:

(Def.2) $c^{\rm op} = c$.

Let us consider C, and let c be an object of $C^{\rm op}$. The functor ${}^{\rm op}c$ yielding an object of C is defined by:

(Def.3) ${}^{\rm op}c = c^{\rm op}$.

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One can prove the following three propositions:

(3) For every object c of C holds $(c^{\mathrm{op}})^{\mathrm{op}} = c$.

(4) For every object c of C holds ${}^{\mathrm{op}}(c^{\mathrm{op}}) = c$.

(5) For every object c of C^{op} holds $({}^{\mathrm{op}}c)^{\mathrm{op}} = c$.

Let us consider C, and let f be a morphism of C. The functor f^{op} yields a morphism of C^{op} and is defined as follows:

(Def.4) $f^{\mathrm{op}} = f$.

Let us consider C, and let f be a morphism of C^{op}. The functor ${}^{\mathrm{op}}f$ yields a morphism of C and is defined by:

(Def.5) ${}^{\mathrm{op}}f = f^{\mathrm{op}}$.

One can prove the following propositions:

(6) For every morphism f of C holds $(f^{\mathrm{op}})^{\mathrm{op}} = f$.

(7) For every morphism f of C holds ${}^{\mathrm{op}}(f^{\mathrm{op}}) = f$.

(8) For every morphism f of C^{op} holds $({}^{\mathrm{op}}f)^{\mathrm{op}} = f$.

(9) For every morphism f of C holds $\mathrm{dom}(f^{\mathrm{op}}) = \mathrm{cod}\, f$ and $\mathrm{cod}(f^{\mathrm{op}}) = \mathrm{dom}\, f$.

(10) For every morphism f of C^{op} holds $\mathrm{dom}\, {}^{\mathrm{op}}f = \mathrm{cod}\, f$ and $\mathrm{cod}\, {}^{\mathrm{op}}f = \mathrm{dom}\, f$.

(11) For every morphism f of C holds $(\mathrm{dom}\, f)^{\mathrm{op}} = \mathrm{cod}(f^{\mathrm{op}})$ and $(\mathrm{cod}\, f)^{\mathrm{op}} = \mathrm{dom}(f^{\mathrm{op}})$.

(12) For every morphism f of C^{op} holds ${}^{\mathrm{op}}\mathrm{dom}\, f = \mathrm{cod}\, {}^{\mathrm{op}}f$ and ${}^{\mathrm{op}}\mathrm{cod}\, f = \mathrm{dom}\, {}^{\mathrm{op}}f$.

(13) For all objects a, b of C and for every morphism f of C holds $f \in \mathrm{hom}(a, b)$ if and only if $f^{\mathrm{op}} \in \mathrm{hom}(b^{\mathrm{op}}, a^{\mathrm{op}})$.

(14) For all objects a, b of C^{op} and for every morphism f of C^{op} holds $f \in \mathrm{hom}(a, b)$ if and only if ${}^{\mathrm{op}}f \in \mathrm{hom}({}^{\mathrm{op}}b, {}^{\mathrm{op}}a)$.

(15) For all objects a, b of C and for every morphism f from a to b such that $\mathrm{hom}(a, b) \neq \emptyset$ holds f^{op} is a morphism from b^{op} to a^{op}.

(16) For all objects a, b of C^{op} and for every morphism f from a to b such that $\mathrm{hom}(a, b) \neq \emptyset$ holds ${}^{\mathrm{op}}f$ is a morphism from ${}^{\mathrm{op}}b$ to ${}^{\mathrm{op}}a$.

(17) For all morphisms f, g of C such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds $(g \cdot f)^{\mathrm{op}} = f^{\mathrm{op}} \cdot g^{\mathrm{op}}$.

(18) For all morphisms f, g of C such that $\mathrm{cod}(g^{\mathrm{op}}) = \mathrm{dom}(f^{\mathrm{op}})$ holds $(g \cdot f)^{\mathrm{op}} = f^{\mathrm{op}} \cdot g^{\mathrm{op}}$.

(19) For all morphisms f, g of C^{op} such that $\mathrm{dom}\, g = \mathrm{cod}\, f$ holds ${}^{\mathrm{op}}(g \cdot f) = {}^{\mathrm{op}}f \cdot {}^{\mathrm{op}}g$.

(20) For all objects a, b, c of C and for every morphism f from a to b and for every morphism g from b to c such that $\mathrm{hom}(a, b) \neq \emptyset$ and $\mathrm{hom}(b, c) \neq \emptyset$ holds $(g \cdot f)^{\mathrm{op}} = f^{\mathrm{op}} \cdot g^{\mathrm{op}}$.

(21) For every object a of C holds $\mathrm{id}_a^{\mathrm{op}} = \mathrm{id}_{a^{\mathrm{op}}}$.

(22) For every object a of C^{op} holds ${}^{\mathrm{op}}(\mathrm{id}_a) = \mathrm{id}_{({}^{\mathrm{op}}a)}$.

(23) For every morphism f of C holds f^{op} is monic if and only if f is epi.

(24) For every morphism f of C holds f^{op} is epi if and only if f is monic.

(25) For every morphism f of C holds f^{op} is invertible if and only if f is invertible.

(26) For every object c of C holds c is an initial object if and only if c^{op} is a terminal object.

(27) For every object c of C holds c^{op} is an initial object if and only if c is a terminal object.

Let us consider C, B, and let S be a function from the morphisms of C^{op} into the morphisms of B. The functor ${}_*S$ yields a function from the morphisms of C into the morphisms of B and is defined by:

(Def.6) for every morphism f of C holds $({}_*S)(f) = S(f^{\mathrm{op}})$.

One can prove the following propositions:

(28) For every function S from the morphisms of C^{op} into the morphisms of B and for every morphism f of C^{op} holds $({}_*S)({}^{\mathrm{op}}f) = S(f)$.

(29) For every functor S from C^{op} to B and for every object c of C holds $(\mathrm{Obj}\,{}_*S)(c) = (\mathrm{Obj}\,S)(c^{\mathrm{op}})$.

(30) For every functor S from C^{op} to B and for every object c of C^{op} holds $(\mathrm{Obj}\,{}_*S)({}^{\mathrm{op}}c) = (\mathrm{Obj}\,S)(c)$.

Let us consider C, D. A function from the morphisms of C into the morphisms of D is called a contravariant functor from C into D if it satisfies the conditions (Def.7).

(Def.7) (i) For every object c of C there exists an object d of D such that $\mathrm{it}(\mathrm{id}_c) = \mathrm{id}_d$,

(ii) for every morphism f of C holds $\mathrm{it}(\mathrm{id}_{\mathrm{dom}\,f}) = \mathrm{id}_{\mathrm{cod}(\mathrm{it}(f))}$ and $\mathrm{it}(\mathrm{id}_{\mathrm{cod}\,f}) = \mathrm{id}_{\mathrm{dom}(\mathrm{it}(f))}$,

(iii) for all morphisms f, g of C such that $\mathrm{dom}\,g = \mathrm{cod}\,f$ holds $\mathrm{it}(g \cdot f) = \mathrm{it}(f) \cdot \mathrm{it}(g)$.

The following propositions are true:

(31) For every contravariant functor S from C into D and for every object c of C and for every object d of D such that $S(\mathrm{id}_c) = \mathrm{id}_d$ holds $(\mathrm{Obj}\,S)(c) = d$.

(32) For every contravariant functor S from C into D and for every object c of C holds $S(\mathrm{id}_c) = \mathrm{id}_{(\mathrm{Obj}\,S)(c)}$.

(33) For every contravariant functor S from C into D and for every morphism f of C holds $(\mathrm{Obj}\,S)(\mathrm{dom}\,f) = \mathrm{cod}(S(f))$ and $(\mathrm{Obj}\,S)(\mathrm{cod}\,f) = \mathrm{dom}(S(f))$.

(34) For every contravariant functor S from C into D and for all morphisms f, g of C such that $\mathrm{dom}\,g = \mathrm{cod}\,f$ holds $\mathrm{dom}(S(f)) = \mathrm{cod}(S(g))$.

(35) For every functor S from C^{op} to B holds ${}_*S$ is a contravariant functor from C into B.

(36) For every contravariant functor S_1 from C into B and for every contravariant functor S_2 from B into D holds $S_2 \cdot S_1$ is a functor from C to D.

(37) For every contravariant functor S from $C^{\rm op}$ into B and for every object c of C holds $(\mathrm{Obj}\,{}_*S)(c) = (\mathrm{Obj}\,S)(c^{\rm op})$.

(38) For every contravariant functor S from $C^{\rm op}$ into B and for every object c of $C^{\rm op}$ holds $(\mathrm{Obj}\,{}_*S)({}^{\rm op}c) = (\mathrm{Obj}\,S)(c)$.

(39) For every contravariant functor S from $C^{\rm op}$ into B holds ${}_*S$ is a functor from C to B.

We now define two new functors. Let us consider C, B, and let S be a function from the morphisms of C into the morphisms of B. The functor ${}^*S$ yielding a function from the morphisms of $C^{\rm op}$ into the morphisms of B is defined as follows:

(Def.8) for every morphism f of $C^{\rm op}$ holds $({}^*S)(f) = S({}^{\rm op}f)$.

The functor S^* yields a function from the morphisms of C into the morphisms of $B^{\rm op}$ and is defined by:

(Def.9) for every morphism f of C holds $S^*(f) = S(f)^{\rm op}$.

The following propositions are true:

(40) For every function S from the morphisms of C into the morphisms of B and for every morphism f of C holds $({}^*S)(f^{\rm op}) = S(f)$.

(41) For every functor S from C to B and for every object c of $C^{\rm op}$ holds $(\mathrm{Obj}\,{}^*S)(c) = (\mathrm{Obj}\,S)({}^{\rm op}c)$.

(42) For every functor S from C to B and for every object c of C holds $(\mathrm{Obj}\,{}^*S)(c^{\rm op}) = (\mathrm{Obj}\,S)(c)$.

(43) For every functor S from C to B and for every object c of C holds $(\mathrm{Obj}(S^*))(c) = (\mathrm{Obj}\,S)(c)^{\rm op}$.

(44) For every contravariant functor S from C into B and for every object c of $C^{\rm op}$ holds $(\mathrm{Obj}\,{}^*S)(c) = (\mathrm{Obj}\,S)({}^{\rm op}c)$.

(45) For every contravariant functor S from C into B and for every object c of C holds $(\mathrm{Obj}\,{}^*S)(c^{\rm op}) = (\mathrm{Obj}\,S)(c)$.

(46) For every contravariant functor S from C into B and for every object c of C holds $(\mathrm{Obj}(S^*))(c) = (\mathrm{Obj}\,S)(c)^{\rm op}$.

(47) For every function F from the morphisms of C into the morphisms of D and for every morphism f of C holds $({}^*F)^*(f^{\rm op}) = F(f)^{\rm op}$.

(48) For every function S from the morphisms of C into the morphisms of D holds ${}_*{}^*S = S$.

(49) For every function S from the morphisms of $C^{\rm op}$ into the morphisms of D holds ${}^*{}_*S = S$.

(50) For every function S from the morphisms of C into the morphisms of D holds $({}^*S)^* = {}^*(S^*)$.

(51) For every function S from the morphisms of C into the morphisms of D holds $(S^*)^* = S$.

(52) For every function S from the morphisms of C into the morphisms of D holds ${}^*({}^*S) = S$.

(53) For every function S from the morphisms of C into the morphisms of B and for every function T from the morphisms of B into the morphisms of D holds ${}^*(T \cdot S) = T \cdot {}^*S$.

(54) For every function S from the morphisms of C into the morphisms of B and for every function T from the morphisms of B into the morphisms of D holds $(T \cdot S)^* = T^* \cdot S$.

(55) For every function F_1 from the morphisms of C into the morphisms of B and for every function F_2 from the morphisms of B into the morphisms of D holds $({}^*(F_2 \cdot F_1))^* = ({}^*F_2)^* \cdot ({}^*F_1)^*$.

(56) For every contravariant functor S from C into D holds ${}^*S$ is a functor from $C^{\rm op}$ to D.

(57) For every contravariant functor S from C into D holds S^* is a functor from C to $D^{\rm op}$.

(58) For every functor S from C to D holds ${}^*S$ is a contravariant functor from $C^{\rm op}$ into D.

(59) For every functor S from C to D holds S^* is a contravariant functor from C into $D^{\rm op}$.

(60) For every contravariant functor S_1 from C into B and for every functor S_2 from B to D holds $S_2 \cdot S_1$ is a contravariant functor from C into D.

(61) For every functor S_1 from C to B and for every contravariant functor S_2 from B into D holds $S_2 \cdot S_1$ is a contravariant functor from C into D.

(62) For every functor F from C to D and for every object c of C holds $({\rm Obj}(({}^*F)^*))(c^{\rm op}) = ({\rm Obj}\, F)(c)^{\rm op}$.

(63) For every contravariant functor F from C into D and for every object c of C holds $({\rm Obj}(({}^*F)^*))(c^{\rm op}) = ({\rm Obj}\, F)(c)^{\rm op}$.

(64) For every functor F from C to D holds $({}^*F)^*$ is a functor from $C^{\rm op}$ to $D^{\rm op}$.

(65) For every contravariant functor F from C into D holds $({}^*F)^*$ is a contravariant functor from $C^{\rm op}$ into $D^{\rm op}$.

We now define two new functors. Let us consider C. The functor ${\rm id}^{\rm op}(C)$ yielding a contravariant functor from C into $C^{\rm op}$ is defined as follows:

(Def.10) ${\rm id}^{\rm op}(C) = {\rm id}_C^*$.

The functor ${}^{\rm op}{\rm id}(C)$ yielding a contravariant functor from $C^{\rm op}$ into C is defined as follows:

(Def.11) ${}^{\rm op}{\rm id}(C) = {}^*({\rm id}_C)$.

One can prove the following propositions:

(66) For every morphism f of C holds ${\rm id}^{\rm op}(C)(f) = f^{\rm op}$.

(67) For every object c of C holds $(\mathrm{Obj}\,\mathrm{id}^{\mathrm{op}}(C))(c) = c^{\mathrm{op}}$.

(68) For every morphism f of C^{op} holds $({}^{\mathrm{op}}\mathrm{id}(C))(f) = {}^{\mathrm{op}}f$.

(69) For every object c of C^{op} holds $(\mathrm{Obj}\,{}^{\mathrm{op}}\mathrm{id}(C))(c) = {}^{\mathrm{op}}c$.

(70) For every function S from the morphisms of C into the morphisms of D holds ${}^{*}S = S \cdot {}^{\mathrm{op}}\mathrm{id}(C)$ and $S^{*} = \mathrm{id}^{\mathrm{op}}(D) \cdot S$.

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Mostowski's Fundamental Operations - Part II

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Summary. The article consists of two parts. The first part is translation of chapter II.3 of [18]. A section of $D_H(a)$ determined by f (symbolically $S_H(a,f)$) and a notion of predicative closure of a class are defined. It is proved that if following assumptions are satisfied: (o) $A = \bigcup_\xi A_\xi$, (i) $A_\xi \subset A_\eta$ for $\xi < \eta$, (ii) $A_\lambda = \bigcup_{\xi<\lambda} A_\lambda$ (λ is a limit number), (iii) $A_\xi \in A$, (iv) A_ξ is transitive, (v) $(x, y \in A) \to (x \cap y \in A)$, (vi) A is predicatively closed, then the axiom of power sets and the axiom of substitution are valid in A. The second part is continuation of [17]. It is proved that if a non-void transitive class is closed under the operations $A_1 - A_7$ then it is predicatively closed. At last sufficient criteria for a class to be a model of ZF-theory are formulated: if A_ξ satisfies o – iv and A is closed under the operations $A_1 - A_7$ then A is a model of ZF.

MML Identifier: ZF_FUND2.

The papers [21], [20], [3], [14], [15], [16], [8], [6], [7], [9], [12], [2], [1], [5], [11], [13], [19], [4], [10], [22], and [17] provide the terminology and notation for this paper. For simplicity we adopt the following rules: H will denote a ZF-formula, M, E will denote non-empty sets, e will denote an element of E, m will denote an element of M, v will denote a function from VAR into M, and f will denote a function from VAR into E. Let us consider H, M, v. The functor $\mathrm{S}_v(H)$ yields a subset of M and is defined by:

(Def.1) (i) $\mathrm{S}_v(H) = \{m : M, v(\frac{x_0}{m}) \models H\}$ if $x_0 \in \mathrm{Free}\, H$,
(ii) $\mathrm{S}_v(H) = \emptyset$, otherwise.

Let us consider M. We say that M is predicatively closed if and only if:

(Def.2) for all H, E, f such that $E \in M$ holds $\mathrm{S}_f(H) \in M$.

We now state the proposition

(1) If E is transitive, then $\mathrm{S}_{f(\frac{x_1}{e})}(\forall_{x_2}(x_2 \epsilon (x_0) \Rightarrow x_2 \epsilon (x_1))) = E \cap 2^e$.

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For simplicity we adopt the following convention: W denotes a universal class, Y denotes a subclass of W, a, b denote ordinals of W, and L denotes a transfinite sequence of non-empty sets from W. We now state several propositions:

(2) If for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$ and for every a holds $L(a) \in \bigcup L$ and $L(a)$ is transitive and $\bigcup L$ is predicatively closed, then $\bigcup L \models$ the axiom of power sets.

(3) Suppose that
(i) $\omega \in W$,
(ii) for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$,
(iii) for every a such that $a \neq \mathbf{0}$ and a is a limit ordinal number holds $L(a) = \bigcup(L \upharpoonright a)$,
(iv) for every a holds $L(a) \in \bigcup L$ and $L(a)$ is transitive,
(v) $\bigcup L$ is predicatively closed.
Then for every H such that $\{x_0, x_1, x_2\}$ misses Free H holds $\bigcup L \models$ the axiom of substitution for H.

(4) $\mathrm{S}_v(H) = \{m : \{\langle \mathbf{0}, m \rangle\} \cup (v \cdot \mathrm{decode}) \upharpoonright (\mathrm{code}(\mathrm{Free}\, H) \setminus \{\mathbf{0}\}) \in \mathrm{D}_M(H)\}$.

(5) If Y is closed w.r.t. A1-A7 and Y is transitive, then Y is predicatively closed.

(6) Suppose that
(i) $\omega \in W$,
(ii) for all a, b such that $a \in b$ holds $L(a) \subseteq L(b)$,
(iii) for every a such that $a \neq \mathbf{0}$ and a is a limit ordinal number holds $L(a) = \bigcup(L \upharpoonright a)$,
(iv) for every a holds $L(a) \in \bigcup L$ and $L(a)$ is transitive,
(v) $\bigcup L$ is closed w.r.t. A1-A7.
Then $\bigcup L$ is a model of ZF.

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Fundamental Types of Metric Affine Spaces

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Summary. We distinguish in the class of metric affine spaces some fundamental types of them. First we can assume the underlying affine space to satisfy classical affine configurational axiom; thus we come to Pappian, Desarguesian, Moufangian, and translation spaces. Next we distinguish the spaces satisfying theorem on three perpendiculars and the homogeneous spaces; these properties directly refer to some axioms involving orthogonality. Some known relationships between the introduced classes of structures are established. We also show that the commonly investigated models of metric affine geometry constructed in a real linear space with the help of a symmetric bilinear form belong to all the classes introduced in the paper.

MML Identifier: EUCLMETR.

The papers [1], [3], [5], [6], [2], [4], [7], [8], and [9] provide the notation and terminology for this paper. A metric affine space is Euclidean if:

(Def.1) for all elements a, b, c, d of the points of it such that $a,b \perp c,d$ and $b,c \perp a,d$ holds $b,d \perp a,c$.

A metric affine space is Pappian if:

(Def.2) the affine reduct of it is Pappian.

A metric affine space is Desarguesian if:

(Def.3) the affine reduct of it is Desarguesian.

A metric affine space is Fanoian if:

(Def.4) the affine reduct of it is Fanoian.

A metric affine space is Moufangian if:

(Def.5) the affine reduct of it is Moufangian.

A metric affine space is translation if:

(Def.6) the affine reduct of it is translation.

A metric affine space is homogeneous if it satisfies the condition (Def.7).

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(Def.7) Let o, a, a_1, b, b_1, c, c_1 be elements of the points of it . Then if $o, a \perp o, a_1$ and $o, b \perp o, b_1$ and $o, c \perp o, c_1$ and $a, b \perp a_1, b_1$ and $a, c \perp a_1, c_1$ and $o, c \nparallel o, a$ and $o, a \nparallel o, b$, then $b, c \perp b_1, c_1$.

In the sequel M_1 denotes a metric affine plane and M_2 denotes a metric affine space. The following propositions are true:

(1) For all elements a, b, c of the points of M_2 such that not $\mathbf{L}(a, b, c)$ holds $a \neq b$ and $b \neq c$ and $a \neq c$.

(2) For all elements a, b, c, d of the points of M_1 and for every subset K of the points of M_1 such that $a, b \perp K$ and $c, d \perp K$ holds $a, b \parallel c, d$ and $a, b \parallel d, c$.

(3) For all elements a, b of the points of M_1 and for all subsets A, K of the points of M_1 such that $a \neq b$ but $a, b \perp K$ or $b, a \perp K$ but $a, b \perp A$ or $b, a \perp A$ holds $K \parallel A$.

(4) For all elements x, y, z of the points of M_2 such that $\mathbf{L}(x, y, z)$ holds $\mathbf{L}(x, z, y)$ and $\mathbf{L}(y, x, z)$ and $\mathbf{L}(y, z, x)$ and $\mathbf{L}(z, x, y)$ and $\mathbf{L}(z, y, x)$.

(5) For all elements a, b, c of the points of M_1 such that not $\mathbf{L}(a, b, c)$ there exists an element d of the points of M_1 such that $d, a \perp b, c$ and $d, b \perp a, c$.

(6) For all elements a, b, c, d_1, d_2 of the points of M_1 such that not $\mathbf{L}(a, b, c)$ and $d_1, a \perp b, c$ and $d_1, b \perp a, c$ and $d_2, a \perp b, c$ and $d_2, b \perp a, c$ holds $d_1 = d_2$.

(7) For all elements a, b, c, d of the points of M_1 such that $a, b \perp c, d$ and $b, c \perp a, d$ and $\mathbf{L}(a, b, c)$ holds $a = c$ or $a = b$ or $b = c$.

(8) M_1 is Euclidean if and only if theorem on three perpendiculars holds in M_1.

(9) M_1 is homogeneous if and only if othogonal verion of Desargues Axiom holds in M_1.

(10) M_1 is Pappian if and only if Pappos Axiom holds in M_1.

(11) M_1 is Desarguesian if and only if Desargues Axiom holds in M_1.

(12) M_1 is Moufangian if and only if trapezium variant of Desargues Axiom holds in M_1.

(13) M_1 is translation if and only if minor Desargues Axiom holds in M_1.

(14) If M_1 is homogeneous, then M_1 is Desarguesian.

(15) If M_1 is Euclidean Desarguesian, then M_1 is Pappian.

We adopt the following rules: V will denote a real linear space and w, y, u, v will denote vectors of V. The following propositions are true:

(16) Let o, c, c_1, a, a_1, a_2 be elements of the points of M_1. Then if not $\mathbf{L}(o, c, a)$ and $o \neq c_1$ and $o, c \perp o, c_1$ and $o, a \perp o, a_1$ and $o, a \perp o, a_2$ and $c, a \perp c_1, a_1$ and $c, a \perp c_1, a_2$, then $a_1 = a_2$.

(17) For all elements o, c, c_1, a of the points of M_1 such that not $\mathbf{L}(o, c, a)$ and $o \neq c_1$ and $o, c \perp o, c_1$ there exists an element a_1 of the points of M_1 such that $o, a \perp o, a_1$ and $c, a \perp c_1, a_1$.

(18) Let a, b be real numbers. Suppose w, y span the space and $0_V \neq u$ and $0_V \neq v$ and u, v are orthogonal w.r.t. w, y and $u = a \cdot w + b \cdot y$. Then there exists a real number c such that $c \neq 0$ and $v = c \cdot b \cdot w + (-c \cdot a) \cdot y$.

(19) Suppose w, y span the space and $0_V \neq u$ and $0_V \neq v$ and u, v are orthogonal w.r.t. w, y. Then there exists a real number c such that for all real numbers a, b holds $a \cdot w + b \cdot y$, $c \cdot b \cdot w + (-c \cdot a) \cdot y$ are orthogonal w.r.t. w, y and $(a \cdot w + b \cdot y) - u$, $(c \cdot b \cdot w + (-c \cdot a) \cdot y) - v$ are orthogonal w.r.t. w, y.

(20) If w, y span the space and $M_1 = \mathbf{AMSp}(V, w, y)$, then for an arbitrary x holds x is a vector of V if and only if x is an element of the points of M_1.

(21) If w, y span the space and $M_1 = \mathbf{AMSp}(V, w, y)$, then LIN holds in M_1.

(22) Suppose w, y span the space and $M_1 = \mathbf{AMSp}(V, w, y)$. Let o, a, a_1, b, b_1, c, c_1 be elements of the points of M_1. Suppose $o, a \perp o, a_1$ and $o, b \perp o, b_1$ and $o, c \perp o, c_1$ and $a, b \perp a_1, b_1$ and $a, c \perp a_1, c_1$ and $o, c \nparallel o, a$ and $o, a \nparallel o, b$ and $o = a_1$. Then $b, c \perp b_1, c_1$.

(23) If w, y span the space and $M_1 = \mathbf{AMSp}(V, w, y)$, then M_1 is homogeneous.

The following proposition is true

(24) If w, y span the space and $M_1 = \mathbf{AMSp}(V, w, y)$, then M_1 is a metric affine plane.

Let M_1 be an Pappian metric affine plane. Then the affine reduct of M_1 is a Pappian affine plane.

Let M_1 be a Desarguesian metric affine plane. Then the affine reduct of M_1 is a Desarguesian affine plane.

Let M_1 be a Moufangian metric affine plane. Then the affine reduct of M_1 is a Moufangian affine plane.

Let M_1 be a translation metric affine plane. Then the affine reduct of M_1 is an translation affine plane.

Let M_1 be an Fanoian metric affine plane. Then the affine reduct of M_1 is a Fanoian affine plane.

Let M_1 be a homogeneous metric affine plane. Then the affine reduct of M_1 is an Desarguesian affine plane.

Let M_1 be a Euclidean Desarguesian metric affine plane. Then
the affine reduct of M_1
is a Pappian affine plane.

Let M_1 be an Pappian metric affine space. Then the affine reduct of M_1 is a Pappian affine space.

Let M_1 be a Desarguesian metric affine space. Then the affine reduct of M_1 is a Desarguesian affine space.

Let M_1 be an Moufangian metric affine space. Then the affine reduct of M_1 is an Moufangian affine space.

Let M_1 be a translation metric affine space. Then the affine reduct of M_1 is a translation affine space.

Let M_1 be a Fanoian metric affine space. Then the affine reduct of M_1 is a Fanoian affine space.

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Filters - Part II. Quotient Lattices Modulo Filters and Direct Product of Two Lattices

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Summary. Binary and unary operation preserving binary relations and quotients of those operations modulo equivalence relations are introduced. It is shown that the quotients inherit some important properties (commutativity, associativity, distributivity, ect.). Based on it the quotient (also called factor) lattice modulo filter (ie. modulo the equivalence relation w.r.t the filter) is introduced. Similarly, some properties of the direct product of two binary (unary) operations are presented and then the direct product of two lattices is introduced. Besides, the heredity of distributivity, modularity, completeness, etc., for the product of lattices is also shown. Finally, the concept of isomorphic lattices is introduced, and it is shown that every Boolean lattice B is isomorphic with the direct product of the factor lattice $B/[a]$ and the lattice latt$[a]$, where a is an element of B.

MML Identifier: `FILTER_1`.

The notation and terminology used in this paper are introduced in the following papers: [11], [5], [6], [13], [4], [8], [12], [9], [2], [3], [7], [14], [1], and [10]. Let L be a lattice structure. An element of L is an element of the carrier of L.

For simplicity we adopt the following convention: L, L_1, L_2 denote lattices, F_1, F_2 denote filters of L, p, q denote elements of L, p_1, q_1 denote elements of L_1, p_2, q_2 denote elements of L_2, x, x_1, y, y_1 are arbitrary, D, D_1, D_2 denote non-empty sets, R denotes a binary relation, R_1 denotes an equivalence relation of D, a, b, d denote elements of D, a_1, b_1 denote elements of D_1, a_2, b_2 denote elements of D_2, B denotes a boolean lattice, F_3 denotes a filter of B, I denotes an implicative lattice, F_4 denotes a filter of I, i, i_1, i_2, j, j_1, j_2, k denote elements of I, f_1, g_1 denote binary operations on D_1, and f_2, g_2 denote binary operations on D_2. One can prove the following two propositions:

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(1) $F_1 \cap F_2$ is a filter of L.

(2) If $[p] = [q]$, then $p = q$.

Let us consider L, F_1, F_2. Then $F_1 \cap F_2$ is a filter of L.

We now define two new modes. Let us consider D, R. A unary operation on D is called a unary R-congruent operation on D if:

(Def.1) for all elements x, y of D such that $\langle x, y\rangle \in R$ holds $\langle \mathrm{it}(x), \mathrm{it}(y)\rangle \in R$.

A binary operation on D is called a binary R-congruent operation on D if:

(Def.2) for all elements x_1, y_1, x_2, y_2 of D such that $\langle x_1, y_1\rangle \in R$ and $\langle x_2, y_2\rangle \in R$ holds $\langle \mathrm{it}(x_1, x_2), \mathrm{it}(y_1, y_2)\rangle \in R$.

In the sequel F, G denote binary R_1-congruent operations on D. We now define two new modes. Let us consider D, and let R be an equivalence relation of D. A unary operation on R is a unary R-congruent operation on D.

A binary operation on R is a binary R-congruent operation on D.

Then $\mathrm{Classes}\, R$ is an non-empty subset of 2^D.

Let X be a set, and let S be a non-empty subset of 2^X. We see that the element of S is a subset of X.

Let us consider D, and let R be an equivalence relation of D, and let d be an element of D. Then $[d]_R$ is an element of $\mathrm{Classes}\, R$.

Let us consider D, and let R be an equivalence relation of D, and let u be a unary operation on D. Let us assume that u is a unary R-congruent operation on D. The functor $u_{/R}$ yielding a unary operation on $\mathrm{Classes}\, R$ is defined as follows:

(Def.3) for all x, y such that $x \in \mathrm{Classes}\, R$ and $y \in x$ holds $u_{/R}(x) = [u(y)]_R$.

Let us consider D, and let R be an equivalence relation of D, and let b be a binary operation on D. Let us assume that b is a binary R-congruent operation on D. The functor $b_{/R}$ yields a binary operation on $\mathrm{Classes}\, R$ and is defined by:

(Def.4) for all x, y, x_1, y_1 such that $x \in \mathrm{Classes}\, R$ and $y \in \mathrm{Classes}\, R$ and $x_1 \in x$ and $y_1 \in y$ holds $b_{/R}(x, y) = [b(x_1, y_1)]_R$.

We now state the proposition

(3) $F_{/R_1}([a]_{R_1}, [b]_{R_1}) = [F(a, b)]_{R_1}$.

The following propositions are true:

(4) If F is commutative, then $F_{/R_1}$ is commutative.

(5) If F is associative, then $F_{/R_1}$ is associative.

(6) If d is a left unity w.r.t. F, then $[d]_{R_1}$ is a left unity w.r.t. $F_{/R_1}$.

(7) If d is a right unity w.r.t. F, then $[d]_{R_1}$ is a right unity w.r.t. $F_{/R_1}$.

(8) If d is a unity w.r.t. F, then $[d]_{R_1}$ is a unity w.r.t. $F_{/R_1}$.

(9) If F is left distributive w.r.t. G, then $F_{/R_1}$ is left distributive w.r.t. $G_{/R_1}$.

(10) If F is right distributive w.r.t. G, then $F_{/R_1}$ is right distributive w.r.t. $G_{/R_1}$.

(11) If F is distributive w.r.t. G, then $F_{/R_1}$ is distributive w.r.t. $G_{/R_1}$.

(12) If F absorbs G, then $F_{/R_1}$ absorbs $G_{/R_1}$.

(13) The join operation of I is a binary $\equiv_{F_4}$-congruent operation on the carrier of I.

(14) The meet operation of I is a binary $\equiv_{F_4}$-congruent operation on the carrier of I.

Let L be a lattice, and let F be a filter of L. Let us assume that L is an implicative lattice. The functor $L_{/F}$ yields a lattice and is defined as follows:

(Def.5) for every equivalence relation R of the carrier of L such that $R = \equiv_F$ holds $L_{/F} = \langle \mathrm{Classes}\, R$, (the join operation of $L)_{/R}$, (the meet operation of $L)_{/R}\rangle$.

Let L be a lattice, and let F be a filter of L, and let a be an element of L. Let us assume that L is an implicative lattice. The functor $a_{/F}$ yielding an element of $L_{/F}$ is defined as follows:

(Def.6) for every equivalence relation R of the carrier of L such that $R = \equiv_F$ holds $a_{/F} = [a]_R$.

Next we state several propositions:

(15) $i_{/F_4} \sqcup j_{/F_4} = (i \sqcup j)_{/F_4}$ and $i_{/F_4} \sqcap j_{/F_4} = (i \sqcap j)_{/F_4}$.

(16) $i_{/F_4} \sqsubseteq j_{/F_4}$ if and only if $i \Rightarrow j \in F_4$.

(17) $i \sqcap j \Rightarrow k = i \Rightarrow (j \Rightarrow k)$.

(18) If I is a lower bound lattice, then $I_{/F_4}$ is a lower bound lattice and $\perp_{I_{/F_4}} = (\perp_I)_{/F_4}$.

(19) $I_{/F_4}$ is an upper bound lattice and $\top_{I_{/F_4}} = (\top_I)_{/F_4}$.

(20) $I_{/F_4}$ is an implicative lattice.

(21) $B_{/F_3}$ is a boolean lattice.

Let D_1, D_2 be non-empty sets, and let f_1 be a binary operation on D_1, and let f_2 be a binary operation on D_2. Then $|{:}f_1, f_2{:}|$ is a binary operation on $[{:}\, D_1, D_2\, {:}]$.

We now state the proposition

(22) $|{:}f_1, f_2{:}|(\langle a_1, a_2\rangle, \langle b_1, b_2\rangle) = \langle f_1(a_1, b_1), f_2(a_2, b_2)\rangle$.

One can prove the following propositions:

(23) f_1 is commutative and f_2 is commutative if and only if $|{:}f_1, f_2{:}|$ is commutative.

(24) f_1 is associative and f_2 is associative if and only if $|{:}f_1, f_2{:}|$ is associative.

(25) a_1 is a left unity w.r.t. f_1 and a_2 is a left unity w.r.t. f_2 if and only if $\langle a_1, a_2\rangle$ is a left unity w.r.t. $|{:}f_1, f_2{:}|$.

(26) a_1 is a right unity w.r.t. f_1 and a_2 is a right unity w.r.t. f_2 if and only if $\langle a_1, a_2\rangle$ is a right unity w.r.t. $|{:}f_1, f_2{:}|$.

(27) a_1 is a unity w.r.t. f_1 and a_2 is a unity w.r.t. f_2 if and only if $\langle a_1, a_2\rangle$ is a unity w.r.t. $|{:}f_1, f_2{:}|$.

(28) f_1 is left distributive w.r.t. g_1 and f_2 is left distributive w.r.t. g_2 if and only if $|{:}f_1, f_2{:}|$ is left distributive w.r.t. $|{:}g_1, g_2{:}|$.

(29) f_1 is right distributive w.r.t. g_1 and f_2 is right distributive w.r.t. g_2 if and only if $|{:}f_1, f_2{:}|$ is right distributive w.r.t. $|{:}g_1, g_2{:}|$.

(30) f_1 is distributive w.r.t. g_1 and f_2 is distributive w.r.t. g_2 if and only if $|{:}f_1, f_2{:}|$ is distributive w.r.t. $|{:}g_1, g_2{:}|$.

(31) f_1 absorbs g_1 and f_2 absorbs g_2 if and only if $|{:}f_1, f_2{:}|$ absorbs $|{:}g_1, g_2{:}|$.

Let L_1, L_2 be lattice structures. The functor $[{:} L_1, L_2 {:}]$ yielding a lattice structure is defined by:

(Def.7) $[{:} L_1, L_2 {:}] = \langle [{:}$ the carrier of L_1, the carrier of $L_2 {:}], |{:}$ the join operation of L_1, the join operation of $L_2{:}|, |{:}$ the meet operation of L_1, the meet operation of $L_2{:}|\rangle$.

Let L be a lattice. The functor $\mathrm{LattRel}(L)$ yields a binary relation and is defined as follows:

(Def.8) $\mathrm{LattRel}(L) = \{\langle p, q\rangle : p \sqsubseteq q\}$, where p ranges over elements of the carrier of L, and q ranges over elements of the carrier of L.

We now state two propositions:

(32) $\langle p, q\rangle \in \mathrm{LattRel}(L)$ if and only if $p \sqsubseteq q$.

(33) $\mathrm{dom}\,\mathrm{LattRel}(L) =$ the carrier of L and $\mathrm{rng}\,\mathrm{LattRel}(L) =$ the carrier of L and $\mathrm{field}\,\mathrm{LattRel}(L) =$ the carrier of L.

Let L_1, L_2 be lattices. We say that L_1 and L_2 are isomorphic if and only if:

(Def.9) $\mathrm{LattRel}(L_1)$ and $\mathrm{LattRel}(L_2)$ are isomorphic.

Let us notice that the predicate introduced above is reflexive and symmetric. Then $[{:} L_1, L_2 {:}]$ is a lattice.

Next we state two propositions:

(34) For all lattices L_1, L_2, L_3 such that L_1 and L_2 are isomorphic and L_2 and L_3 are isomorphic holds L_1 and L_3 are isomorphic.

(35) For all L_1, L_2 being lattice structures such that $[{:} L_1, L_2 {:}]$ is a lattice holds L_1 is a lattice and L_2 is a lattice.

Let L_1, L_2 be lattices, and let a be an element of L_1, and let b be an element of L_2. Then $\langle a, b\rangle$ is an element of $[{:} L_1, L_2 {:}]$.

The following propositions are true:

(36) $\langle p_1, p_2\rangle \sqcup \langle q_1, q_2\rangle = \langle p_1 \sqcup q_1, p_2 \sqcup q_2\rangle$ and $\langle p_1, p_2\rangle \sqcap \langle q_1, q_2\rangle = \langle p_1 \sqcap q_1, p_2 \sqcap q_2\rangle$.

(37) $\langle p_1, p_2\rangle \sqsubseteq \langle q_1, q_2\rangle$ if and only if $p_1 \sqsubseteq q_1$ and $p_2 \sqsubseteq q_2$.

(38) L_1 is a modular lattice and L_2 is a modular lattice if and only if $[{:} L_1, L_2 {:}]$ is a modular lattice.

(39) L_1 is a distributive lattice and L_2 is a distributive lattice if and only if $[{:} L_1, L_2 {:}]$ is a distributive lattice.

(40) L_1 is a lower bound lattice and L_2 is a lower bound lattice if and only if $[{:} L_1, L_2 {:}]$ is a lower bound lattice.

(41) L_1 is an upper bound lattice and L_2 is an upper bound lattice if and only if $[\!: L_1, L_2 :\!]$ is an upper bound lattice.

(42) L_1 is a bound lattice and L_2 is a bound lattice if and only if $[\!: L_1, L_2 :\!]$ is a bound lattice.

(43) If L_1 is a lower bound lattice and L_2 is a lower bound lattice, then $\bot_{[\!: L_1, L_2 :\!]} = \langle \bot_{L_1}, \bot_{L_2} \rangle$.

(44) If L_1 is an upper bound lattice and L_2 is an upper bound lattice, then $\top_{[\!: L_1, L_2 :\!]} = \langle \top_{L_1}, \top_{L_2} \rangle$.

(45) If L_1 is a bound lattice and L_2 is a bound lattice, then p_1 is a complement of q_1 and p_2 is a complement of q_2 if and only if $\langle p_1, p_2 \rangle$ is a complement of $\langle q_1, q_2 \rangle$.

(46) L_1 is a complemented lattice and L_2 is a complemented lattice if and only if $[\!: L_1, L_2 :\!]$ is a complemented lattice.

(47) L_1 is a boolean lattice and L_2 is a boolean lattice if and only if $[\!: L_1, L_2 :\!]$ is a boolean lattice.

(48) L_1 is an implicative lattice and L_2 is an implicative lattice if and only if $[\!: L_1, L_2 :\!]$ is an implicative lattice.

(49) $[\!: L_1, L_2 :\!]^\circ = [\!: L_1{}^\circ, L_2{}^\circ :\!]$.

(50) $[\!: L_1, L_2 :\!]$ and $[\!: L_2, L_1 :\!]$ are isomorphic.

We follow the rules: B will be a boolean lattice and a, b, c, d will be elements of B. One can prove the following propositions:

(51) $a \Leftrightarrow b = a \sqcap b \sqcup a^{\rm c} \sqcap b^{\rm c}$.

(52) $(a \Rightarrow b)^{\rm c} = a \sqcap b^{\rm c}$ and $(a \Leftrightarrow b)^{\rm c} = a \sqcap b^{\rm c} \sqcup a^{\rm c} \sqcap b$ and $(a \Leftrightarrow b)^{\rm c} = a \Leftrightarrow b^{\rm c}$ and $(a \Leftrightarrow b)^{\rm c} = a^{\rm c} \Leftrightarrow b$.

(53) If $a \Leftrightarrow b = a \Leftrightarrow c$, then $b = c$.

(54) $a \Leftrightarrow (a \Leftrightarrow b) = b$.

(55) $i \sqcup j \Rightarrow i = j \Rightarrow i$ and $i \Rightarrow i \sqcap j = i \Rightarrow j$.

(56) $i \Rightarrow j \sqsubseteq i \Rightarrow j \sqcup k$ and $i \Rightarrow j \sqsubseteq i \sqcap k \Rightarrow j$ and $i \Rightarrow j \sqsubseteq i \Rightarrow k \sqcup j$ and $i \Rightarrow j \sqsubseteq k \sqcap i \Rightarrow j$.

(57) $(i \Rightarrow k) \sqcap (j \Rightarrow k) \sqsubseteq i \sqcup j \Rightarrow k$.

(58) $(i \Rightarrow j) \sqcap (i \Rightarrow k) \sqsubseteq i \Rightarrow j \sqcap k$.

(59) If $i_1 \Leftrightarrow i_2 \in F_4$ and $j_1 \Leftrightarrow j_2 \in F_4$, then $i_1 \sqcup j_1 \Leftrightarrow i_2 \sqcup j_2 \in F_4$ and $i_1 \sqcap j_1 \Leftrightarrow i_2 \sqcap j_2 \in F_4$.

(60) If $i \in [k]_{\equiv_{F_4}}$ and $j \in [k]_{\equiv_{F_4}}$, then $i \sqcup j \in [k]_{\equiv_{F_4}}$ and $i \sqcap j \in [k]_{\equiv_{F_4}}$.

(61) $c \sqcup (c \Leftrightarrow d) \in [c]_{\equiv_{[d]}}$ and for every b such that $b \in [c]_{\equiv_{[d]}}$ holds $b \sqsubseteq c \sqcup (c \Leftrightarrow d)$.

(62) B and $[\!: B_{/[a]}, \mathbb{L}_{[a]} :\!]$ are isomorphic.

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Shear Theorems and Their Role in Affine Geometry

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Summary. Investigations on affine shear theorems, major and minor, direct and indirect. We prove logical relationships which hold between these statements and between them and other classical affine configurational axioms (eg. minor and major Pappus Axiom, Desargues Axioms et al.). For the shear, Desargues and Pappus Axioms formulated in terms of metric affine spaces we prove that they are equivalent to corresponding statements formulated in terms of affine reduct of the given space.

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The terminology and notation used in this paper have been introduced in the following papers: [2], [4], [1], [3], [6], [7], and [5]. We follow a convention: X will be an affine plane, o, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 will be elements of the points of X, and M, N will be subsets of the points of X. Let us consider X. We say that X satisfies minor Scherungssatz if and only if the condition (Def.1) is satisfied.

(Def.1) Given a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, M, N. Suppose that
- (i) $M \parallel N$,
- (ii) $a_1 \in M$,
- (iii) $a_3 \in M$,
- (iv) $b_1 \in M$,
- (v) $b_3 \in M$,
- (vi) $a_2 \in N$,
- (vii) $a_4 \in N$,
- (viii) $b_2 \in N$,
- (ix) $b_4 \in N$,
- (x) $a_4 \notin M$,

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(xi) $a_2 \notin M$,
(xii) $b_2 \notin M$,
(xiii) $b_4 \notin M$,
(xiv) $a_1 \notin N$,
(xv) $a_3 \notin N$,
(xvi) $b_1 \notin N$,
(xvii) $b_3 \notin N$,
(xviii) $a_3, a_2 \parallel b_3, b_2$,
(xix) $a_2, a_1 \parallel b_2, b_1$,
(xx) $a_1, a_4 \parallel b_1, b_4$.
Then $a_3, a_4 \parallel b_3, b_4$.

Let us consider X. We say that X satisfies major Scherungssatz if and only if the condition (Def.2) is satisfied.

(Def.2) Given o, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, M, N. Suppose that
(i) M is a line,
(ii) N is a line,
(iii) $o \in M$,
(iv) $o \in N$,
(v) $a_1 \in M$,
(vi) $a_3 \in M$,
(vii) $b_1 \in M$,
(viii) $b_3 \in M$,
(ix) $a_2 \in N$,
(x) $a_4 \in N$,
(xi) $b_2 \in N$,
(xii) $b_4 \in N$,
(xiii) $a_4 \notin M$,
(xiv) $a_2 \notin M$,
(xv) $b_2 \notin M$,
(xvi) $b_4 \notin M$,
(xvii) $a_1 \notin N$,
(xviii) $a_3 \notin N$,
(xix) $b_1 \notin N$,
(xx) $b_3 \notin N$,
(xxi) $a_3, a_2 \parallel b_3, b_2$,
(xxii) $a_2, a_1 \parallel b_2, b_1$,
(xxiii) $a_1, a_4 \parallel b_1, b_4$.
Then $a_3, a_4 \parallel b_3, b_4$.

Let us consider X. We say that X satisfies Scherungssatz if and only if the condition (Def.3) is satisfied.

(Def.3) Given a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, M, N. Suppose that
(i) M is a line,
(ii) N is a line,
(iii) $a_1 \in M$,

(iv) $a_3 \in M$,
(v) $b_1 \in M$,
(vi) $b_3 \in M$,
(vii) $a_2 \in N$,
(viii) $a_4 \in N$,
(ix) $b_2 \in N$,
(x) $b_4 \in N$,
(xi) $a_4 \notin M$,
(xii) $a_2 \notin M$,
(xiii) $b_2 \notin M$,
(xiv) $b_4 \notin M$,
(xv) $a_1 \notin N$,
(xvi) $a_3 \notin N$,
(xvii) $b_1 \notin N$,
(xviii) $b_3 \notin N$,
(xix) $a_3, a_2 \parallel b_3, b_2$,
(xx) $a_2, a_1 \parallel b_2, b_1$,
(xxi) $a_1, a_4 \parallel b_1, b_4$.

Then $a_3, a_4 \parallel b_3, b_4$.

Let us consider X. We say that X satisfies Scherungssatz* if and only if the condition (Def.4) is satisfied.

(Def.4) Given a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, M, N. Suppose that
(i) M is a line,
(ii) N is a line,
(iii) $a_1 \in M$,
(iv) $a_3 \in M$,
(v) $b_2 \in M$,
(vi) $b_4 \in M$,
(vii) $a_2 \in N$,
(viii) $a_4 \in N$,
(ix) $b_1 \in N$,
(x) $b_3 \in N$,
(xi) $a_4 \notin M$,
(xii) $a_2 \notin M$,
(xiii) $b_1 \notin M$,
(xiv) $b_3 \notin M$,
(xv) $a_1 \notin N$,
(xvi) $a_3 \notin N$,
(xvii) $b_2 \notin N$,
(xviii) $b_4 \notin N$,
(xix) $a_3, a_2 \parallel b_3, b_2$,
(xx) $a_2, a_1 \parallel b_2, b_1$,
(xxi) $a_1, a_4 \parallel b_1, b_4$.

Then $a_3, a_4 \parallel b_3, b_4$.

Let us consider X. We say that X satisfies minor Scherungssatz* if and only if the condition (Def.5) is satisfied.

(Def.5) Given a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, M, N. Suppose that
(i) $M \parallel N$,
(ii) $a_1 \in M$,
(iii) $a_3 \in M$,
(iv) $b_2 \in M$,
(v) $b_4 \in M$,
(vi) $a_2 \in N$,
(vii) $a_4 \in N$,
(viii) $b_1 \in N$,
(ix) $b_3 \in N$,
(x) $a_4 \notin M$,
(xi) $a_2 \notin M$,
(xii) $b_1 \notin M$,
(xiii) $b_3 \notin M$,
(xiv) $a_1 \notin N$,
(xv) $a_3 \notin N$,
(xvi) $b_2 \notin N$,
(xvii) $b_4 \notin N$,
(xviii) $a_3, a_2 \parallel b_3, b_2$,
(xix) $a_2, a_1 \parallel b_2, b_1$,
(xx) $a_1, a_4 \parallel b_1, b_4$.
Then $a_3, a_4 \parallel b_3, b_4$.

Let us consider X. We say that X satisfies major Scherungssatz* if and only if the condition (Def.6) is satisfied.

(Def.6) Given o, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, M, N. Suppose that
(i) M is a line,
(ii) N is a line,
(iii) $o \in M$,
(iv) $o \in N$,
(v) $a_1 \in M$,
(vi) $a_3 \in M$,
(vii) $b_2 \in M$,
(viii) $b_4 \in M$,
(ix) $a_2 \in N$,
(x) $a_4 \in N$,
(xi) $b_1 \in N$,
(xii) $b_3 \in N$,
(xiii) $a_4 \notin M$,
(xiv) $a_2 \notin M$,
(xv) $b_1 \notin M$,
(xvi) $b_3 \notin M$,
(xvii) $a_1 \notin N$,

(xviii) $a_3 \notin N$,
(xix) $b_2 \notin N$,
(xx) $b_4 \notin N$,
(xxi) $a_3, a_2 \parallel b_3, b_2$,
(xxii) $a_2, a_1 \parallel b_2, b_1$,
(xxiii) $a_1, a_4 \parallel b_1, b_4$.

Then $a_3, a_4 \parallel b_3, b_4$.

Next we state a number of propositions:

(1) X satisfies Scherungssatz* if and only if X satisfies minor Scherungssatz* and X satisfies major Scherungssatz*.

(2) X satisfies Scherungssatz if and only if X satisfies minor Scherungssatz and X satisfies major Scherungssatz.

(3) If X satisfies minor Scherungssatz*, then X satisfies minor Scherungssatz.

(4) If X satisfies major Scherungssatz*, then X satisfies major Scherungssatz.

(5) If X satisfies Scherungssatz*, then X satisfies Scherungssatz.

(6) If X satisfies **des**, then X satisfies minor Scherungssatz.

(7) If X satisfies **DES**, then X satisfies major Scherungssatz.

(8) X satisfies **DES** if and only if X satisfies Scherungssatz.

(9) X satisfies **pap** if and only if X satisfies minor Scherungssatz*.

(10) X satisfies **PAP** if and only if X satisfies major Scherungssatz*.

(11) X satisfies **PPAP** if and only if X satisfies Scherungssatz*.

(12) If X satisfies major Scherungssatz*, then X satisfies minor Scherungssatz *.

In the sequel X denotes a metric affine plane. We now state several propositions:

(13) The affine reduct of X satisfies Scherungssatz if and only if Scherungssatz holds in X.

(14) trapezium variant of Desargues Axiom holds in X if and only if the affine reduct of X satisfies **TDES**.

(15) The affine reduct of X satisfies **des** if and only if minor Desargues Axiom holds in X.

(16) Pappos Axiom holds in X if and only if the affine reduct of X satisfies **PAP**.

(17) Desargues Axiom holds in X if and only if the affine reduct of X satisfies **DES**.

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