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Classical and Non–classical Pasch Configurations in Ordered Affine Planes [1]

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Summary. Several configuration axioms, which are commonly called in the literature "Pasch Axioms" are introduced; three of them were investigated by Szmielew and concern invariantability of the betweenness relation under parallel projections, and two other were introduced by Tarski. It is demonstrated that they all are consequences of the trapezium axiom, adopted to characterize ordered affine spaces.

MML Identifier: `PASCH`.

The papers [1] and [2] provide the notation and terminology for this paper. We adopt the following rules: OAS will be an ordered affine space and a, a', b, b', c, c', d, d_1, d_2, p, p', x, y, z, t, u will be elements of the points of OAS. Let us consider OAS. We say that OAS satisfies inner invariancy of betweenness relation under parallel projections if and only if:

for all a, b, c, d, p such that not $\mathbf{L}(p,b,c)$ and $\mathbf{B}(b,p,a)$ and $\mathbf{L}(p,c,d)$ and $b,c \parallel d,a$ holds $\mathbf{B}(c,p,d)$.

We now state the proposition

(1) For every OAS holds OAS satisfies inner invariancy of betweenness relation under parallel projections if and only if for all a, b, c, d, p such that not $\mathbf{L}(p,b,c)$ and $\mathbf{B}(b,p,a)$ and $\mathbf{L}(p,c,d)$ and $b,c \parallel d,a$ holds $\mathbf{B}(c,p,d)$.

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Let us consider OAS. We say that OAS satisfies outer invariancy of betweenness relation under parallel projections if and only if:

for all a, b, c, d, p such that $\mathbf{B}(p,b,c)$ and $\mathbf{L}(p,a,d)$ and $a,b \parallel c,d$ and not $\mathbf{L}(p,a,b)$ holds $\mathbf{B}(p,a,d)$.

We now state the proposition

(2) For every OAS holds OAS satisfies outer invariancy of betweenness relation under parallel projections if and only if for all a, b, c, d, p such that $\mathbf{B}(p,b,c)$ and $\mathbf{L}(p,a,d)$ and $a,b \parallel c,d$ and not $\mathbf{L}(p,a,b)$ holds $\mathbf{B}(p,a,d)$.

Let us consider OAS. We say that OAS satisfies general invariancy of betweenness relation under parallel projections if and only if:

for all a, b, c, a', b', c' such that not $\mathbf{L}(a,b,a')$ and $a,a' \parallel b,b'$ and $a,a' \parallel c,c'$ and $\mathbf{B}(a,b,c)$ and $\mathbf{L}(a',b',c')$ holds $\mathbf{B}(a',b',c')$.

We now state the proposition

(3) For every OAS holds OAS satisfies general invariancy of betweenness relation under parallel projections if and only if for all a, b, c, a', b', c' such that not $\mathbf{L}(a,b,a')$ and $a,a' \parallel b,b'$ and $a,a' \parallel c,c'$ and $\mathbf{B}(a,b,c)$ and $\mathbf{L}(a',b',c')$ holds $\mathbf{B}(a',b',c')$.

Let us consider OAS. We say that OAS satisfies outer form of Pasch' Axiom if and only if:

for all a, b, c, d, x, y such that $\mathbf{B}(a,b,d)$ and $\mathbf{B}(b,x,c)$ and not $\mathbf{L}(a,b,c)$ there exists y such that $\mathbf{B}(a,y,c)$ and $\mathbf{B}(y,x,d)$.

The following proposition is true

(4) For every OAS holds OAS satisfies outer form of Pasch' Axiom if and only if for all a, b, c, d, x, y such that $\mathbf{B}(a,b,d)$ and $\mathbf{B}(b,x,c)$ and not $\mathbf{L}(a,b,c)$ there exists y such that $\mathbf{B}(a,y,c)$ and $\mathbf{B}(y,x,d)$.

Let us consider OAS. We say that OAS satisfies inner form of Pasch' Axiom if and only if:

for all a, b, c, d, x, y such that $\mathbf{B}(a,b,d)$ and $\mathbf{B}(a,x,c)$ and not $\mathbf{L}(a,b,c)$ there exists y such that $\mathbf{B}(b,y,c)$ and $\mathbf{B}(x,y,d)$.

The following proposition is true

(5) For every OAS holds OAS satisfies inner form of Pasch' Axiom if and only if for all a, b, c, d, x, y such that $\mathbf{B}(a,b,d)$ and $\mathbf{B}(a,x,c)$ and not $\mathbf{L}(a,b,c)$ there exists y such that $\mathbf{B}(b,y,c)$ and $\mathbf{B}(x,y,d)$.

Let us consider OAS. We say that OAS satisfies Fano Axiom if and only if:

for all a, b, c, d such that $a,b \upuparrows c,d$ and $a,c \upuparrows b,d$ and not $\mathbf{L}(a,b,c)$ there exists x such that $\mathbf{B}(a,x,d)$ and $\mathbf{B}(b,x,c)$.

We now state a number of propositions:

(6) For every OAS holds OAS satisfies Fano Axiom if and only if for all a, b, c, d such that $a,b \upuparrows c,d$ and $a,c \upuparrows b,d$ and not $\mathbf{L}(a,b,c)$ there exists x such that $\mathbf{B}(a,x,d)$ and $\mathbf{B}(b,x,c)$.

(7) If $b,p \upuparrows p,c$ and $p \neq c$ and $b \neq p$, then there exists d such that $a,p \upuparrows p,d$ and $a,b \parallel c,d$ and $c \neq d$ and $p \neq d$.

(8) If $p, b \upuparrows p, c$ and $p \neq c$ and $b \neq p$, then there exists d such that $p, a \upuparrows p, d$ and $a, b \parallel c, d$ and $c \neq d$.

(9) If $p, b \parallel p, c$ and $p \neq b$, then there exists d such that $p, a \parallel p, d$ and $a, b \parallel c, d$.

(10) If $z, x \parallel x, t$ and $x \neq z$, then there exists u such that $y, x \parallel x, u$ and $y, z \parallel t, u$.

(11) If not $\mathbf{L}(p, a, b)$ and $\mathbf{L}(p, b, c)$ and $\mathbf{L}(p, a, d_1)$ and $\mathbf{L}(p, a, d_2)$ and $a, b \parallel c, d_1$ and $a, b \parallel c, d_2$, then $d_1 = d_2$.

(12) If not $\mathbf{L}(a, b, c)$ and $a, b \parallel c, d_1$ and $a, b \parallel c, d_2$ and $a, c \parallel b, d_1$ and $a, c \parallel b, d_2$, then $d_1 = d_2$.

(13) If not $\mathbf{L}(p, b, c)$ and $\mathbf{B}(b, p, a)$ and $\mathbf{L}(p, c, d)$ and $b, c \parallel d, a$, then $\mathbf{B}(c, p, d)$.

(14) *OAS* satisfies inner invariancy of betweenness relation under parallel projections.

(15) If $\mathbf{B}(p, b, c)$ and $\mathbf{L}(p, a, d)$ and $a, b \parallel c, d$ and not $\mathbf{L}(p, a, b)$, then $\mathbf{B}(p, a, d)$.

(16) *OAS* satisfies outer invariancy of betweenness relation under parallel projections.

(17) If not $\mathbf{L}(a, b, a')$ and $a, a' \parallel b, b'$ and $a, a' \parallel c, c'$ and $\mathbf{B}(a, b, c)$ and $\mathbf{L}(a', b', c')$, then $\mathbf{B}(a', b', c')$.

(18) *OAS* satisfies general invariancy of betweenness relation under parallel projections.

(19) If not $\mathbf{L}(p, a, b)$ and $a, p \upuparrows p, a'$ and $b, p \upuparrows p, b'$ and $a, b \parallel a', b'$, then $a, b \upuparrows b', a'$.

(20) If not $\mathbf{L}(p, a, a')$ and $p, a \upuparrows p, b$ and $p, a' \upuparrows p, b'$ and $a, a' \parallel b, b'$, then $a, a' \upuparrows b, b'$.

(21) If not $\mathbf{L}(p, a, b)$ and $p, a \parallel b, c$ and $p, b \parallel a, c$, then $p, a \upuparrows b, c$ and $p, b \upuparrows a, c$.

(22) If $\mathbf{B}(p, c, b)$ and $c, d \upuparrows b, a$ and $p, d \upuparrows p, a$ and not $\mathbf{L}(p, a, b)$ and $p \neq c$, then $\mathbf{B}(p, d, a)$.

(23) If $\mathbf{B}(p, d, a)$ and $c, d \upuparrows b, a$ and $p, c \upuparrows p, b$ and not $\mathbf{L}(p, a, b)$ and $p \neq c$, then $\mathbf{B}(p, c, b)$.

(24) If not $\mathbf{L}(p, a, b)$ and $p, b \upuparrows p, c$ and $b, a \upuparrows c, d$ and $\mathbf{L}(a, p, d)$ and $p \neq d$, then not $\mathbf{B}(a, p, d)$.

(25) If $p, b \upuparrows p, c$ and $b \neq p$, then there exists x such that $p, a \upuparrows p, x$ and $b, a \upuparrows c, x$.

(26) If $\mathbf{B}(p, c, b)$, then there exists x such that $\mathbf{B}(p, x, a)$ and $b, a \upuparrows c, x$.

(27) If $p \neq b$ and $\mathbf{B}(p, b, c)$, then there exists x such that $\mathbf{B}(p, a, x)$ and $b, a \upuparrows c, x$.

(28) If not $\mathbf{L}(p, a, b)$ and $\mathbf{B}(p, c, b)$, then there exists x such that $\mathbf{B}(p, x, a)$ and $a, b \upuparrows x, c$.

(29) There exists x such that $a, x \upuparrows b, c$ and $a, b \upuparrows x, c$.

(30) If $a, b \upuparrows c, d$ and not $\mathbf{L}(a, b, c)$, then there exists x such that $\mathbf{B}(a, x, d)$ and $\mathbf{B}(b, x, c)$.

(31) If $a, b \upuparrows c, d$ and $a, c \upuparrows b, d$ and not $\mathbf{L}(a, b, c)$, then there exists x such that $\mathbf{B}(a, x, d)$ and $\mathbf{B}(b, x, c)$.

(32) *OAS* satisfies Fano Axiom.

(33) If $a, b \parallel c, d$ and $a, c \parallel b, d$ and not $\mathbf{L}(a, b, c)$, then there exists x such that $\mathbf{L}(x, a, d)$ and $\mathbf{L}(x, b, c)$.

(34) If $a, b \parallel c, d$ and $a, c \parallel b, d$ and not $\mathbf{L}(a, b, c)$ and $\mathbf{L}(p, a, d)$ and $\mathbf{L}(p, b, c)$, then not $\mathbf{L}(p, a, b)$.

(35) If $\mathbf{B}(a, b, d)$ and $\mathbf{B}(b, x, c)$ and not $\mathbf{L}(a, b, c)$, then there exists y such that $\mathbf{B}(a, y, c)$ and $\mathbf{B}(y, x, d)$.

(36) *OAS* satisfies outer form of Pasch' Axiom.

(37) If $\mathbf{B}(a, b, d)$ and $\mathbf{B}(a, x, c)$ and not $\mathbf{L}(a, b, c)$, then there exists y such that $\mathbf{B}(b, y, c)$ and $\mathbf{B}(x, y, d)$.

(38) *OAS* satisfies inner form of Pasch' Axiom.

(39) If $\mathbf{B}(p, a, b)$ and $p, a \upuparrows p', a'$ and not $\mathbf{L}(p, a, p')$ and $\mathbf{L}(p', a', b')$ and $p, p' \upuparrows a, a'$ and $p, p' \upuparrows b, b'$, then $\mathbf{B}(p', a', b')$.

References

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