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Ordinal Arithmetics

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Summary. At the beginning the article contains some auxiliary theorems concerning the constructors defined in papers [1] and [2]. Next simple properties of addition and multiplication of ordinals are shown, e.g. associativity of addition. Addition and multiplication of a transfinite sequence of ordinals and a ordinal are also introduced here. The goal of the article is the proof that the distributivity of multiplication wrt addition and the associativity of multiplication hold. Additionally new binary functors of ordinals are introduced: subtraction, exact division, and remainder and some of their basic properties are presented.

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The notation and terminology used here are introduced in the following papers: [5], [3], [1], [4], and [2]. For simplicity we adopt the following convention: fi, psi denote sequences of ordinal numbers, A, B, C, D denote ordinal numbers, X, Y denote sets, and x is arbitrary. We now state a number of propositions:

(1) $X \subseteq \operatorname{succ} X$.

(2) If $\operatorname{succ} X \subseteq Y$, then $X \subseteq Y$.

(3) If $\operatorname{succ} A \subseteq B$, then $A \in B$.

(4) $A \subseteq B$ if and only if $\operatorname{succ} A \subseteq \operatorname{succ} B$.

(5) $A \in B$ if and only if $\operatorname{succ} A \in \operatorname{succ} B$.

(6) If $X \subseteq A$, then $\bigcup X$ is an ordinal number.

(7) $\bigcup(\mathrm{On}\, X)$ is an ordinal number.

(8) If $X \subseteq A$, then $\mathrm{On}\, X = X$.

(9) $\mathrm{On}\{A\} = \{A\}$.

(10) If $A \neq \mathbf{0}$, then $\mathbf{0} \in A$.

(11) $\inf A = \mathbf{0}$.

(12) $\inf\{A\} = A$.

(13) If $X \subseteq A$, then $\bigcap X$ is an ordinal number.

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Let us consider x. Let us assume that x is an ordinal number. The functor x (as an ordinal) yielding an ordinal number, is defined as follows:

x (as an ordinal) $= x$.

The following proposition is true

(14) If x is an ordinal number, then x (as an ordinal) $= x$.

Let us consider A, B. Then $A \cup B$ is an ordinal number. Then $A \cap B$ is an ordinal number.

We now state a number of propositions:

(15) $A \cup B = A$ or $A \cup B = B$.

(16) $A \cap B = A$ or $A \cap B = B$.

(17) If $A \in \mathbf{1}$, then $A = \mathbf{0}$.

(18) $\mathbf{1} = \{\mathbf{0}\}$.

(19) If $A \subseteq \mathbf{1}$, then $A = \mathbf{0}$ or $A = \mathbf{1}$.

(20) If $A \subseteq B$ or $A \in B$ but $C \in D$, then $A + C \in B + D$.

(21) If $A \subseteq B$ and $C \subseteq D$, then $A + C \subseteq B + D$.

(22) If $A \in B$ but $C \subseteq D$ and $D \neq \mathbf{0}$ or $C \in D$, then $A \cdot C \in B \cdot D$.

(23) If $A \subseteq B$ and $C \subseteq D$, then $A \cdot C \subseteq B \cdot D$.

(24) If $B + C = B + D$, then $C = D$.

(25) If $B + C \in B + D$, then $C \in D$.

(26) If $B + C \subseteq B + D$, then $C \subseteq D$.

(27) $A \subseteq A + B$ and $B \subseteq A + B$.

(28) If $A \in B$, then $A \in B + C$ and $A \in C + B$.

(29) If $A + B = \mathbf{0}$, then $A = \mathbf{0}$ and $B = \mathbf{0}$.

(30) If $A \subseteq B$, then there exists C such that $B = A + C$.

(31) If $A \in B$, then there exists C such that $B = A + C$ and $C \neq \mathbf{0}$.

(32) If $A \neq \mathbf{0}$ and A is a limit ordinal number, then $B + A$ is a limit ordinal number.

(33) $(A + B) + C = A + (B + C)$.

(34) If $A \cdot B = \mathbf{0}$, then $A = \mathbf{0}$ or $B = \mathbf{0}$.

(35) If $A \in B$ and $C \neq \mathbf{0}$, then $A \in B \cdot C$ and $A \in C \cdot B$.

(36) If $B \cdot A = C \cdot A$ and $A \neq \mathbf{0}$, then $B = C$.

(37) If $B \cdot A \in C \cdot A$, then $B \in C$.

(38) If $B \cdot A \subseteq C \cdot A$ and $A \neq \mathbf{0}$, then $B \subseteq C$.

(39) If $B \neq \mathbf{0}$, then $A \subseteq A \cdot B$ and $A \subseteq B \cdot A$.

(40) If $A \in B$ and $C \neq \mathbf{0}$, then $A \in B \cdot C$ and $A \in C \cdot B$.

(41) If $A \cdot B = \mathbf{1}$, then $A = \mathbf{1}$ and $B = \mathbf{1}$.

(42) If $A \in B + C$, then $A \in B$ or there exists D such that $D \in C$ and $A = B + D$.

We now define four new functors. Let us consider C, fi. The functor $C + fi$ yields a sequence of ordinal numbers and is defined by:

$\mathrm{dom}(C + fi) = \mathrm{dom}\, fi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $(C + fi)(A) = C + (fi(A))$ (as an ordinal).

The functor $fi + C$ yields a sequence of ordinal numbers and is defined by:

$\mathrm{dom}(fi + C) = \mathrm{dom}\, fi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $(fi + C)(A) = (fi(A))$ (as an ordinal) $+ C$.

The functor $C \cdot fi$ yields a sequence of ordinal numbers and is defined as follows:

$\mathrm{dom}(C \cdot fi) = \mathrm{dom}\, fi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $(C \cdot fi)(A) = C \cdot (fi(A))$ (as an ordinal).

The functor $fi \cdot C$ yields a sequence of ordinal numbers and is defined by:

$\mathrm{dom}(fi \cdot C) = \mathrm{dom}\, fi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $(fi \cdot C)(A) = (fi(A))$ (as an ordinal) $\cdot C$.

The following propositions are true:

(43) $psi = C + fi$ if and only if $\mathrm{dom}\, psi = \mathrm{dom}\, fi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $psi(A) = C + (fi(A))$ (as an ordinal).

(44) $psi = fi + C$ if and only if $\mathrm{dom}\, psi = \mathrm{dom}\, fi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $psi(A) = (fi(A))$ (as an ordinal) $+ C$.

(45) $psi = C \cdot fi$ if and only if $\mathrm{dom}\, psi = \mathrm{dom}\, fi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $psi(A) = C \cdot (fi(A))$ (as an ordinal).

(46) $psi = fi \cdot C$ if and only if $\mathrm{dom}\, psi = \mathrm{dom}\, fi$ and for every A such that $A \in \mathrm{dom}\, fi$ holds $psi(A) = (fi(A))$ (as an ordinal) $\cdot C$.

(47) If $\mathbf{0} \neq \mathrm{dom}\, fi$ and $\mathrm{dom}\, fi = \mathrm{dom}\, psi$ and for all A, B such that $A \in \mathrm{dom}\, fi$ and $B = fi(A)$ holds $psi(A) = C + B$, then $\sup psi = C + \sup fi$.

(48) If A is a limit ordinal number, then $A \cdot B$ is a limit ordinal number.

(49) If $A \in B \cdot C$ and B is a limit ordinal number, then there exists D such that $D \in B$ and $A \in D \cdot C$.

(50) If $\mathbf{0} \neq \mathrm{dom}\, fi$ and $\mathrm{dom}\, fi = \mathrm{dom}\, psi$ and $C \neq \mathbf{0}$ and $\sup fi$ is a limit ordinal number and for all A, B such that $A \in \mathrm{dom}\, fi$ and $B = fi(A)$ holds $psi(A) = B \cdot C$, then $\sup psi = \sup fi \cdot C$.

(51) If $\mathbf{0} \neq \mathrm{dom}\, fi$, then $\sup(C + fi) = C + \sup fi$.

(52) If $\mathbf{0} \neq \mathrm{dom}\, fi$ and $C \neq \mathbf{0}$ and $\sup fi$ is a limit ordinal number, then $\sup(fi \cdot C) = \sup fi \cdot C$.

(53) If $B \neq \mathbf{0}$, then $\bigcup(A + B) = A + \bigcup B$.

(54) $(A + B) \cdot C = A \cdot C + B \cdot C$.

(55) If $A \neq \mathbf{0}$, then there exist C, D such that $B = C \cdot A + D$ and $D \in A$.

(56) For all ordinal numbers C_1, D_1, C_2, D_2 such that $C_1 \cdot A + D_1 = C_2 \cdot A + D_2$ and $D_1 \in A$ and $D_2 \in A$ holds $C_1 = C_2$ and $D_1 = D_2$.

(57) If $\mathbf{1} \in B$ and $A \neq \mathbf{0}$ and A is a limit ordinal number, then for every fi such that $\mathrm{dom}\, fi = A$ and for every C such that $C \in A$ holds $fi(C) = C \cdot B$ holds $A \cdot B = \sup fi$.

(58) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$.

We now define two new functors. Let us consider A, B. The functor $A - B$ yields an ordinal number and is defined as follows:

$A = B + (A - B)$ if $B \subseteq A$, $A - B = \mathbf{0}$, otherwise.
The functor $A \div B$ yielding an ordinal number, is defined by:
there exists C such that $A = (A \div B) \cdot B + C$ and $C \in B$ if $B \neq \mathbf{0}$, $A \div B = \mathbf{0}$, otherwise.

Let us consider A, B. The functor $A \bmod B$ yielding an ordinal number, is defined by:
$A \bmod B = A - (A \div B) \cdot B$.

The following propositions are true:

(59) If $A \subseteq B$, then $B = A + (B - A)$.

(60) If $A \in B$, then $B = A + (B - A)$.

(61) If $A \not\subseteq B$, then $B - A = \mathbf{0}$.

(62) If $B \neq \mathbf{0}$, then there exists C such that $A = (A \div B) \cdot B + C$ and $C \in B$.

(63) $A \div \mathbf{0} = \mathbf{0}$.

(64) $A \bmod B = A - (A \div B) \cdot B$.

(65) $(A + B) - A = B$.

(66) If $A \in B$ but $C \subseteq A$ or $C \in A$, then $A - C \in B - C$.

(67) $A - A = \mathbf{0}$.

(68) If $A \in B$, then $B - A \neq \mathbf{0}$ and $\mathbf{0} \in B - A$.

(69) $A - \mathbf{0} = A$ and $\mathbf{0} - A = \mathbf{0}$.

(70) $A - (B + C) = (A - B) - C$.

(71) If $A \subseteq B$, then $C - B \subseteq C - A$.

(72) If $A \subseteq B$, then $A - C \subseteq B - C$.

(73) If $C \neq \mathbf{0}$ and $A \in B + C$, then $A - B \in C$.

(74) If $A + B \in C$, then $B \in C - A$.

(75) $A \subseteq B + (A - B)$.

(76) $A \cdot C - B \cdot C = (A - B) \cdot C$.

(77) $(A \div B) \cdot B \subseteq A$.

(78) $A = (A \div B) \cdot B + (A \bmod B)$.

(79) If $A = B \cdot C + D$ and $D \in C$, then $B = A \div C$ and $D = A \bmod C$.

(80) If $A \in B \cdot C$, then $A \div C \in B$ and $A \bmod C \in C$.

(81) If $B \neq \mathbf{0}$, then $A \cdot B \div B = A$.

(82) $A \cdot B \bmod B = \mathbf{0}$.

(83) $\mathbf{0} \div A = \mathbf{0}$ and $\mathbf{0} \bmod A = \mathbf{0}$ and $A \bmod \mathbf{0} = A$.

(84) $A \div \mathbf{1} = A$ and $A \bmod \mathbf{1} = \mathbf{0}$.

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