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König's Theorem

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Summary. In the article the sum and product of any number of cardinals are introduced and their relationships to addition, multiplication and to other concepts are shown. Then the König's theorem is proved. The theorem that the cardinal of union of increasing family of sets of power less than some cardinal **m** is not greater than **m**, is given too.

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The papers [12], [6], [7], [3], [14], [13], [4], [2], [11], [9], [8], [10], [1], and [5] provide the terminology and notation for this paper. For simplicity we adopt the following rules: A, B are ordinal numbers, K, M, N are cardinal numbers, x, y, z are arbitrary, X, Y, Z, Z_1, Z_2 are sets, n is a natural number, and f, g are functions. A function is said to be a function yielding cardinal numbers if:

for every x such that $x \in \mathrm{dom}\,\mathrm{it}$ holds $\mathrm{it}(x)$ is a cardinal number.

Next we state a proposition

(1) f is a function yielding cardinal numbers if and only if for every x such that $x \in \mathrm{dom}\, f$ holds $f(x)$ is a cardinal number.

In the sequel ff denotes a function yielding cardinal numbers. Let us consider ff, X. Then $ff \restriction X$ is a function yielding cardinal numbers.

Let us consider ff, x. Then $ff(x)$ is a set.

Let us consider X, K. Then $X \longmapsto K$ is a function yielding cardinal numbers.

The following propositions are true:

(2) $ff \restriction X$ is a function yielding cardinal numbers and $X \longmapsto K$ is a function yielding cardinal numbers.

(3) $\square$ is a function yielding cardinal numbers.

The scheme *CF_Lambda* concerns a set $\mathcal{A}$ and a unary functor $\mathcal{F}$ yielding a cardinal number and states that:

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there exists ff such that $\mathrm{dom}\, ff = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ holds $ff(x) = \mathcal{F}(x)$
for all values of the parameters.

We now define four new functors. Let us consider f. The functor $\overline{\overline{f}}$ yields a function yielding cardinal numbers and is defined as follows:

$\mathrm{dom}\, \overline{\overline{f}} = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds $\overline{\overline{f}}(x) = \overline{\overline{[f(x)]}}$.

The functor $\mathrm{disjoin}\, f$ yielding a function, is defined as follows:

$\mathrm{dom}(\mathrm{disjoin}\, f) = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds $(\mathrm{disjoin}\, f)(x) = [\!:[f(x)], \{x\}:\!]$.

The functor $\bigcup f$ yields a set and is defined by:

$\bigcup f = \bigcup(\mathrm{rng}\, f)$.

The functor $\prod f$ yielding a set, is defined by:

$x \in \prod f$ if and only if there exists g such that $x = g$ and $\mathrm{dom}\, g = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds $g(x) \in [f(x)]$.

We now state a number of propositions:

(4) $ff = \overline{\overline{f}}$ if and only if $\mathrm{dom}\, ff = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds $ff(x) = \overline{\overline{[f(x)]}}$.

(5) $g = \mathrm{disjoin}\, f$ if and only if $\mathrm{dom}\, g = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds $g(x) = [\!:[f(x)], \{x\}:\!]$.

(6) $\bigcup f = \bigcup(\mathrm{rng}\, f)$.

(7) $X = \prod f$ if and only if for every x holds $x \in X$ if and only if there exists g such that $x = g$ and $\mathrm{dom}\, g = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds $g(x) \in [f(x)]$.

(8) $\overline{\overline{ff}} = ff$.

(9) $\overline{\overline{\Box}} = \Box$.

(10) $\overline{\overline{X \longmapsto Y}} = X \longmapsto \overline{\overline{Y}}$.

(11) $\mathrm{disjoin}\, \Box = \Box$.

(12) $\mathrm{disjoin}(\{x\} \longmapsto X) = \{x\} \longmapsto [\!:X, \{x\}:\!]$.

(13) If $x \in \mathrm{dom}\, f$ and $y \in \mathrm{dom}\, f$ and $x \neq y$, then $[\mathrm{disjoin}\, f(x)] \cap [\mathrm{disjoin}\, f(y)] = \emptyset$.

(14) $\bigcup \Box = \emptyset$.

(15) $\bigcup(X \longmapsto Y) \subseteq Y$.

(16) If $X \neq \emptyset$, then $\bigcup(X \longmapsto Y) = Y$.

(17) $\bigcup(\{x\} \longmapsto Y) = Y$.

(18) $g \in \prod f$ if and only if $\mathrm{dom}\, g = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds $g(x) \in [f(x)]$.

(19) $\prod \Box = \{\Box\}$.

(20) $Y^X = \prod(X \longmapsto Y)$.

Let us consider x, X. The functor $\pi_x X$ yields a set and is defined by:

$y \in \pi_x X$ if and only if there exists f such that $f \in X$ and $y = f(x)$.

Next we state a number of propositions:

(21) $Y = \pi_x X$ if and only if for every y holds $y \in Y$ if and only if there exists f such that $f \in X$ and $y = f(x)$.

(22) If $x \in \operatorname{dom} f$ and $\prod f \neq \emptyset$, then $\pi_x(\prod f) = f(x)$.

(23) If $f \in X$, then $f(x) \in \pi_x X$.

(24) $\pi_x \emptyset = \emptyset$.

(25) $\pi_x\{g\} = \{g(x)\}$.

(26) $\pi_x\{f, g\} = \{f(x), g(x)\}$.

(27) $\pi_x(X \cup Y) = \pi_x X \cup \pi_x Y$.

(28) $\pi_x(X \cap Y) \subseteq \pi_x X \cap \pi_x Y$.

(29) $\pi_x X \setminus \pi_x Y \subseteq \pi_x(X \setminus Y)$.

(30) $\pi_x X \dot{-} \pi_x Y \subseteq \pi_x(X \dot{-} Y)$.

(31) $\overline{\overline{\pi_x X}} \leq \overline{\overline{X}}$.

(32) If $x \in \bigcup(\operatorname{disjoin} f)$, then there exist y, z such that $x = \langle y, z \rangle$.

(33) $x \in \bigcup(\operatorname{disjoin} f)$ if and only if $x_{\mathbf{2}} \in \operatorname{dom} f$ and $x_{\mathbf{1}} \in [f(x_{\mathbf{2}})]$ and $x = \langle x_{\mathbf{1}}, x_{\mathbf{2}} \rangle$.

(34) If $f \leq g$, then $\operatorname{disjoin} f \leq \operatorname{disjoin} g$.

(35) If $f \leq g$, then $\bigcup f \subseteq \bigcup g$.

(36) $\bigcup(\operatorname{disjoin}(Y \longmapsto X)) = [: X, Y :]$.

(37) $\prod f = \emptyset$ if and only if $\emptyset \in \operatorname{rng} f$.

(38) If $\operatorname{dom} f = \operatorname{dom} g$ and for every x such that $x \in \operatorname{dom} f$ holds $[f(x)] \subseteq [g(x)]$, then $\prod f \subseteq \prod g$.

In the sequel F, G will denote functions yielding cardinal numbers. The following two propositions are true:

(39) For every x such that $x \in \operatorname{dom} F$ holds $\overline{\overline{F(x)}} = F(x)$.

(40) For every x such that $x \in \operatorname{dom} F$ holds $\overline{\overline{[\operatorname{disjoin} F(x)]}} = F(x)$.

We now define two new functors. Let us consider F. The functor $\sum F$ yields a cardinal number and is defined as follows:

$\sum F = \overline{\overline{\bigcup(\operatorname{disjoin} F)}}$.

The functor $\prod F$ yielding a cardinal number, is defined as follows:

$\prod F = \overline{\overline{\prod F}}$.

The following propositions are true:

(41) $\sum F = \overline{\overline{\bigcup(\operatorname{disjoin} F)}}$.

(42) $\prod F = \overline{\overline{\prod F}}$.

(43) If $\operatorname{dom} F = \operatorname{dom} G$ and for every x such that $x \in \operatorname{dom} F$ holds $F(x) \subseteq G(x)$, then $\sum F \leq \sum G$.

(44) $\emptyset \in \operatorname{rng} F$ if and only if $\prod F = \overline{\mathbf{0}}$.

(45) If $\operatorname{dom} F = \operatorname{dom} G$ and for every x such that $x \in \operatorname{dom} F$ holds $F(x) \subseteq G(x)$, then $\prod F \leq \prod G$.

(46) If $F \leq G$, then $\sum F \leq \sum G$.

(47) If $F \leq G$ and $\overline{\mathbf{0}} \notin \operatorname{rng} G$, then $\prod F \leq \prod G$.

(48) $\sum(\emptyset \longmapsto K) = \overline{\mathbf{0}}$.

(49) $\prod(\emptyset \longmapsto K) = \overline{\mathbf{1}}$.

(50) $\sum(\{x\} \longmapsto K) = K$.

(51) $\prod(\{x\} \longmapsto K) = K$.

(52) $\sum(M \longmapsto N) = M \cdot N$.

(53) $\prod(N \longmapsto M) = M^N$.

(54) $\overline{\overline{\bigcup f}} \leq \sum \overline{\overline{f}}$.

(55) $\overline{\overline{\bigcup F}} \leq \sum F$.

(56) If $\operatorname{dom} F = \operatorname{dom} G$ and for every x such that $x \in \operatorname{dom} F$ holds $F(x) \in G(x)$, then $\sum F < \prod G$.

Now we present three schemes. The scheme *FinRegularity* deals with a set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists x such that $x \in \mathcal{A}$ and for every y such that $y \in \mathcal{A}$ and $y \neq x$ holds not $\mathcal{P}[y, x]$

provided the following conditions are fulfilled:

- $\mathcal{A}$ is finite and $\mathcal{A} \neq \emptyset$,
- for all x, y such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, x]$ holds $x = y$,
- for all x, y, z such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

The scheme *MaxFinSetElem* concerns a set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists x such that $x \in \mathcal{A}$ and for every y such that $y \in \mathcal{A}$ holds $\mathcal{P}[x, y]$

provided the following requirements are fulfilled:

- $\mathcal{A}$ is finite and $\mathcal{A} \neq \emptyset$,
- for all x, y holds $\mathcal{P}[x, y]$ or $\mathcal{P}[y, x]$,
- for all x, y, z such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

The scheme *FuncSeparation* deals with a set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a binary predicate $\mathcal{P}$, and states that:

there exists f such that $\operatorname{dom} f = \mathcal{A}$ and for every x such that $x \in \mathcal{A}$ for every y holds $y \in [f(x)]$ if and only if $y \in \mathcal{F}(x)$ and $\mathcal{P}[x, y]$

for all values of the parameters.

We now state several propositions:

(57) $\mathbf{R}_{\mathrm{ord}(n)}$ is finite.

(58) If X is finite, then $\overline{\overline{X}} < \overline{\overline{\omega}}$.

(59) If $\overline{\overline{A}} < \overline{\overline{B}}$, then $A \in B$.

(60) If $\overline{\overline{A}} < M$, then $A \in M$.

(61) Suppose for all Z_1, Z_2 such that $Z_1 \in X$ and $Z_2 \in X$ holds $Z_1 \subseteq Z_2$ or $Z_2 \subseteq Z_1$. Then there exists Y such that $Y \subseteq X$ and $\bigcup Y = \bigcup X$ and for every Z such that $Z \subseteq Y$ and $Z \neq \emptyset$ there exists Z_1 such that $Z_1 \in Z$ and for every Z_2 such that $Z_2 \in Z$ holds $Z_1 \subseteq Z_2$.

(62) If for every Z such that $Z \in X$ holds $\overline{\overline{Z}} < M$ and for all Z_1, Z_2 such that $Z_1 \in X$ and $Z_2 \in X$ holds $Z_1 \subseteq Z_2$ or $Z_2 \subseteq Z_1$, then $\overline{\overline{\bigcup X}} \leq M$.

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