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Semilattice Operations on Finite Subsets

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Summary. In the article we deal with a binary operation that is associative, commutative. We define for such an operation a functor that depends on two more arguments: a finite set of indices and a function indexing elements of the domain of the operation and yields the result of applying the operation to all indexed elements. The definition has a restriction that requires that either the set of indices is non empty or the operation has the unity. We prove theorems describing some properties of the functor introduced. Most of them we prove in two versions depending on which requirement is fulfilled. In the second part we deal with the union of finite sets that enjoys mentioned above properties. We prove analogs of the theorems proved in the first part. We precede the main part of the article with auxiliary theorems related to boolean properties of sets, enumerated sets, finite subsets, and functions. We define a casting function that yields to a set the empty set typed as a finite subset of the set. We prove also two schemes of the induction on finite sets.

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The terminology and notation used in this paper have been introduced in the following articles: [5], [4], [7], [6], [2], [1], and [3]. We adopt the following rules: x, y, z will be arbitrary and X, Y, Z, X', Y' will be sets. The following propositions are true:

(1) If $\{x\} \subseteq \{y\}$, then $x = y$.
(2) $\{x, y, z\} \neq \emptyset$.
(3) $\{x\} \subseteq \{x, y, z\}$.
(4) $\{x, y\} \subseteq \{x, y, z\}$.
(5) If $X \subseteq Y \cup \{x\}$, then $x \in X$ or $X \subseteq Y$.
(6) $x \in X \cup \{y\}$ if and only if $x \in X$ or $x = y$.
(7) If $X \cup Y \subseteq Z$, then $X \subseteq Z$ and $Y \subseteq Z$.

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(8) $X \cup \{x\} \subseteq Y$ if and only if $x \in Y$ and $X \subseteq Y$.

(9) If $X' \cup Y' = X \cup Y$ and X misses X' and Y misses Y', then $X = Y'$ and $Y = X'$.

(10) If $X' \cup Y' = X \cup Y$ and Y misses X' and X misses Y', then $X = X'$ and $Y = Y'$.

(11) For all X, Y for every function f holds $f\,^\circ\,(Y \setminus f^{\,-1}\,X) = f\,^\circ\,Y \setminus X$.

In the sequel X, Y will denote non-empty sets and f will denote a function from X into Y. Next we state two propositions:

(12) For every element x of X holds $x \in f^{\,-1}\,\{f(x)\}$.

(13) For every element x of X holds $f\,^\circ\,\{x\} = \{f(x)\}$.

The scheme *SubsetEx* deals with a constant $\mathcal{A}$ that is a non-empty set and a unary predicate $\mathcal{P}$ and states that:

there exists B being a subset of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ holds $x \in B$ if and only if $\mathcal{P}[x]$

for all values of the parameters.

We now state several propositions:

(14) For every element B of $\operatorname{Fin} X$ for every x such that $x \in B$ holds x is an element of X.

(15) For every element A of $\operatorname{Fin} X$ for every set B for every function f from X into Y such that for every element x of X such that $x \in A$ holds $f(x) \in B$ holds $f\,^\circ\,A \subseteq B$.

(16) For every set X for every element B of $\operatorname{Fin} X$ for every set A such that $A \subseteq B$ holds A is an element of $\operatorname{Fin} X$.

(17) For every element A of $\operatorname{Fin} X$ holds $f\,^\circ\,A$ is an element of $\operatorname{Fin} Y$.

(18) For every element B of $\operatorname{Fin} X$ such that $B \neq \emptyset$ there exists x being an element of X such that $x \in B$.

(19) For every element A of $\operatorname{Fin} X$ such that $f\,^\circ\,A = \emptyset$ holds $A = \emptyset$.

Let X be a set. The functor 0_X yielding an element of $\operatorname{Fin} X$, is defined by:

$0_X = \emptyset$.

One can prove the following proposition

(20) For every set X holds $0_X = \emptyset$.

The arguments of the notions defined below are the following: X which is a non-empty set; A which is a set; f which is a function from X into $\operatorname{Fin} A$; x which is an element of X. Then $f(x)$ is an element of $\operatorname{Fin} A$.

The scheme *FinSubFuncEx* deals with a constant $\mathcal{A}$ that is a non-empty set, a constant $\mathcal{B}$ that is an element of $\operatorname{Fin} \mathcal{A}$ and a binary predicate $\mathcal{P}$ and states that:

there exists f being a function from $\mathcal{A}$ into $\operatorname{Fin} \mathcal{A}$ such that for all elements b, a of $\mathcal{A}$ holds $a \in f(b)$ if and only if $a \in \mathcal{B}$ and $\mathcal{P}[a, b]$

for all values of the parameters.

The arguments of the notions defined below are the following: X which is a non-empty set; F which is a binary operation on X. The predicate F has a unity is defined by:

there exists x being an element of X such that x is a unity w.r.t. F.

We now state three propositions:

(21) For every non-empty set X for every binary operation F on X holds F has a unity if and only if there exists x being an element of X such that x is a unity w.r.t. F.

(22) For every non-empty set X for every binary operation F on X holds F has a unity if and only if $\mathbf{1}_F$ is a unity w.r.t. F.

(23) For every non-empty set X for every binary operation F on X such that F has a unity for every element x of X holds $F(\mathbf{1}_F, x) = x$ and $F(x, \mathbf{1}_F) = x$.

The arguments of the notions defined below are the following: X which is a non-empty set; x which is an element of X. Then $\{x\}$ is an element of $\mathrm{Fin}\, X$. Let y be an element of X. Then $\{x, y\}$ is an element of $\mathrm{Fin}\, X$. Let z be an element of X. Then $\{x, y, z\}$ is an element of $\mathrm{Fin}\, X$.

Now we present three schemes. The scheme *FinSubInd1* concerns a constant $\mathcal{A}$ that is a non-empty set and a unary predicate $\mathcal{P}$ and states that:

for every element B of $\mathrm{Fin}\, \mathcal{A}$ holds $\mathcal{P}[B]$

provided the parameters satisfy the following conditions:

- $\mathcal{P}[0_{\mathcal{A}}]$,
- for every element B' of $\mathrm{Fin}\, \mathcal{A}$ for every element b of $\mathcal{A}$ such that $\mathcal{P}[B']$ and $b \notin B'$ holds $\mathcal{P}[B' \cup \{b\}]$.

The scheme *FinSubInd2* concerns a constant $\mathcal{A}$ that is a non-empty set and a unary predicate $\mathcal{P}$ and states that:

for every element B of $\mathrm{Fin}\, \mathcal{A}$ such that $B \neq \emptyset$ holds $\mathcal{P}[B]$

provided the parameters satisfy the following conditions:

- for every element x of $\mathcal{A}$ holds $\mathcal{P}[\{x\}]$,
- for all elements B_1, B_2 of $\mathrm{Fin}\, \mathcal{A}$ such that $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$ holds if $\mathcal{P}[B_1]$ and $\mathcal{P}[B_2]$, then $\mathcal{P}[B_1 \cup B_2]$.

The scheme *FinSubInd3* concerns a constant $\mathcal{A}$ that is a non-empty set and a unary predicate $\mathcal{P}$ and states that:

for every element B of $\mathrm{Fin}\, \mathcal{A}$ holds $\mathcal{P}[B]$

provided the parameters satisfy the following conditions:

- $\mathcal{P}[0_{\mathcal{A}}]$,
- for every element B' of $\mathrm{Fin}\, \mathcal{A}$ for every element b of $\mathcal{A}$ such that $\mathcal{P}[B']$ holds $\mathcal{P}[B' \cup \{b\}]$.

The arguments of the notions defined below are the following: X which is a non-empty family of sets; Y which is a non-empty set; f which is a function from X into Y; x which is an element of X. Then $f(x)$ is an element of Y.

In the sequel C will be a non-empty set. The arguments of the notions defined below are the following: X, Y which are non-empty sets; F which is a binary operation on Y; B which is an element of $\mathrm{Fin}\, X$; f which is a function from X into Y. Let us assume that $B \neq \emptyset$ or F has a unity and F is commutative and F is associative. The functor $F\text{-}\sum_B f$ yielding an element of Y, is defined by:

there exists G being a function from $\mathrm{Fin}\, X$ into Y such that $F\text{-}\sum_B f = G(B)$ and for every element e of Y such that e is a unity w.r.t. F holds $G(\emptyset) = e$ and for every element x of X holds $G(\{x\}) = f(x)$ and for every element B' of $\operatorname{Fin} X$ such that $B' \subseteq B$ and $B' \neq \emptyset$ for every element x of X such that $x \in B \setminus B'$ holds $G(B' \cup \{x\}) = F(G(B'), f(x))$.

One can prove the following propositions:

(24) Let X, Y be non-empty sets. Let F be a binary operation on Y. Let B be an element of $\operatorname{Fin} X$. Let f be a function from X into Y. Suppose $B \neq \emptyset$ or F has a unity but F is commutative and F is associative. Let IT be an element of Y. Then $IT = F\text{-}\sum_B f$ if and only if there exists G being a function from $\operatorname{Fin} X$ into Y such that $IT = G(B)$ and for every element e of Y such that e is a unity w.r.t. F holds $G(\emptyset) = e$ and for every element x of X holds $G(\{x\}) = f(x)$ and for every element B' of $\operatorname{Fin} X$ such that $B' \subseteq B$ and $B' \neq \emptyset$ for every element x of X such that $x \in B \setminus B'$ holds $G(B' \cup \{x\}) = F(G(B'), f(x))$.

(25) Let X, Y be non-empty sets. Let F be a binary operation on Y. Let B be an element of $\operatorname{Fin} X$. Let f be a function from X into Y. Suppose $B \neq \emptyset$ or F has a unity but F is idempotent and F is commutative and F is associative. Let IT be an element of Y. Then $IT = F\text{-}\sum_B f$ if and only if there exists G being a function from $\operatorname{Fin} X$ into Y such that $IT = G(B)$ and for every element e of Y such that e is a unity w.r.t. F holds $G(\emptyset) = e$ and for every element x of X holds $G(\{x\}) = f(x)$ and for every element B' of $\operatorname{Fin} X$ such that $B' \subseteq B$ and $B' \neq \emptyset$ for every element x of X such that $x \in B$ holds $G(B' \cup \{x\}) = F(G(B'), f(x))$.

For simplicity we follow the rules: X, Y denote non-empty sets, F denotes a binary operation on Y, B denotes an element of $\operatorname{Fin} X$, and f denotes a function from X into Y. Next we state a number of propositions:

(26) If F is commutative and F is associative, then for every element b of X holds $F\text{-}\sum_{\{b\}} f = f(b)$.

(27) If F is idempotent and F is commutative and F is associative, then for all elements a, b of X holds $F\text{-}\sum_{\{a,b\}} f = F(f(a), f(b))$.

(28) If F is idempotent and F is commutative and F is associative, then for all elements a, b, c of X holds $F\text{-}\sum_{\{a,b,c\}} f = F(F(f(a), f(b)), f(c))$.

(29) If F is idempotent and F is commutative and F is associative and $B \neq \emptyset$, then for every element x of X holds $F\text{-}\sum_{B \cup \{x\}} f = F(F\text{-}\sum_B f, f(x))$.

(30) If F is idempotent and F is commutative and F is associative, then for all elements B_1, B_2 of $\operatorname{Fin} X$ such that $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$ holds $F\text{-}\sum_{B_1 \cup B_2} f = F(F\text{-}\sum_{B_1} f, F\text{-}\sum_{B_2} f)$.

(31) If F is commutative and F is associative and F is idempotent, then for every element x of X such that $x \in B$ holds $F(f(x), F\text{-}\sum_B f) = F\text{-}\sum_B f$.

(32) If F is commutative and F is associative and F is idempotent, then for all elements B, C of $\operatorname{Fin} X$ such that $B \neq \emptyset$ and $B \subseteq C$ holds $F(F\text{-}\sum_B f, F\text{-}\sum_C f) = F\text{-}\sum_C f$.

(33) If $B \neq \emptyset$ and F is commutative and F is associative and F is idempotent, then for every element a of Y such that for every element b of X such that $b \in B$ holds $f(b) = a$ holds $F\text{-}\sum_B f = a$.

(34) If F is commutative and F is associative and F is idempotent, then for every element a of Y such that $f \circ B = \{a\}$ holds $F\text{-}\sum_B f = a$.

(35) If F is commutative and F is associative and F is idempotent, then for all functions f, g from X into Y for all elements A, B of $\operatorname{Fin} X$ such that $A \neq \emptyset$ and $f \circ A = g \circ B$ holds $F\text{-}\sum_A f = F\text{-}\sum_B g$.

(36) Let F, G be binary operations on Y. Then if F is idempotent and F is commutative and F is associative and G is distributive w.r.t. F, then for every element B of $\operatorname{Fin} X$ such that $B \neq \emptyset$ for every function f from X into Y for every element a of Y holds $G(a, F\text{-}\sum_B f) = F\text{-}\sum_B(G^{\circ}(a, f))$.

(37) Let F, G be binary operations on Y. Then if F is idempotent and F is commutative and F is associative and G is distributive w.r.t. F, then for every element B of $\operatorname{Fin} X$ such that $B \neq \emptyset$ for every function f from X into Y for every element a of Y holds $G(F\text{-}\sum_B f, a) = F\text{-}\sum_B(G^{\circ}(f, a))$.

The arguments of the notions defined below are the following: A, X, Y which are non-empty sets; f which is a function from X into Y; g which is a function from Y into A. Then $g \cdot f$ is a function from X into A.

The arguments of the notions defined below are the following: X, Y which are non-empty sets; f which is a function from X into Y; A which is an element of $\operatorname{Fin} X$. Then $f \circ A$ is an element of $\operatorname{Fin} Y$.

The following propositions are true:

(38) Let A, X, Y be non-empty sets. Then for every binary operation F on A such that F is idempotent and F is commutative and F is associative for every element B of $\operatorname{Fin} X$ such that $B \neq \emptyset$ for every function f from X into Y for every function g from Y into A holds $F\text{-}\sum_{f \circ B} g = F\text{-}\sum_B(g \cdot f)$.

(39) Suppose F is commutative and F is associative and F is idempotent. Let Z be a non-empty set. Let G be a binary operation on Z. Suppose G is commutative and G is associative and G is idempotent. Let f be a function from X into Y. Then for every function g from Y into Z such that for all elements x, y of Y holds $g(F(x, y)) = G(g(x), g(y))$ for every element B of $\operatorname{Fin} X$ such that $B \neq \emptyset$ holds $g(F\text{-}\sum_B f) = G\text{-}\sum_B(g \cdot f)$.

(40) If F is commutative and F is associative and F has a unity, then for every f holds $F\text{-}\sum_{0_X} f = \mathbf{1}_F$.

(41) If F is idempotent and F is commutative and
F
is associative and F has a unity, then for every element x of X holds $F\text{-}\sum_{B \cup \{x\}} f = F(F\text{-}\sum_B f, f(x))$.

(42) If F is idempotent and F is commutative and F is associative and F has a unity, then for all elements B_1, B_2 of $\operatorname{Fin} X$ holds $F\text{-}\sum_{B_1 \cup B_2} f = F(F\text{-}\sum_{B_1} f, F\text{-}\sum_{B_2} f)$.

(43) If F is commutative and F is associative and F is idempotent and F has a unity, then for all functions f, g from X into Y for all elements A, B of $\operatorname{Fin} X$ such that $f \circ A = g \circ B$ holds $F\text{-}\sum_A f = F\text{-}\sum_B g$.

(44) For all non-empty sets A, X, Y for every binary operation F on A such

that F is idempotent and F is commutative and F is associative and F has a unity for every element B of $\mathrm{Fin}\, X$ for every function f from X into Y for every function g from Y into A holds $F\text{-}\sum_{f^\circ B} g = F\text{-}\sum_B (g \cdot f)$.

(45) Suppose F is commutative and F is associative and F is idempotent and F has a unity. Let Z be a non-empty set. Let G be a binary operation on Z. Suppose G is commutative and G is associative and G is idempotent and G has a unity. Let f be a function from X into Y. Let g be a function from Y into Z. Then if $g(\mathbf{1}_F) = \mathbf{1}_G$ and for all elements x, y of Y holds $g(F(x, y)) = G(g(x), g(y))$, then for every element B of $\mathrm{Fin}\, X$ holds $g(F\text{-}\sum_B f) = G\text{-}\sum_B (g \cdot f)$.

The arguments of the notions defined below are the following: A which is a set; x which is an element of $\mathrm{Fin}\, A$. The functor $@x$ yielding an element of $\mathrm{Fin}\, A$ **qua** a non-empty set, is defined by:

$@x = x$.

The following proposition is true

(46) For every set A for every element x of $\mathrm{Fin}\, A$ holds $@x = x$.

Let A be a set. The functor $\mathrm{FinUnion}_A$ yields a binary operation on $\mathrm{Fin}\, A$ and is defined by:

for all elements x, y of $\mathrm{Fin}\, A$ holds $(\mathrm{FinUnion}_A)(x, y) = @(x \cup y)$.

In the sequel A will denote a set and x, y will denote elements of $\mathrm{Fin}\, A$. One can prove the following propositions:

(47) For every binary operation IT on $\mathrm{Fin}\, A$ holds $IT = \mathrm{FinUnion}_A$ if and only if for all elements x, y of $\mathrm{Fin}\, A$ holds $IT(x, y) = @(x \cup y)$.

(48) $\mathrm{FinUnion}_A(x, y) = x \cup y$.

(49) $\mathrm{FinUnion}_A$ is idempotent.

(50) $\mathrm{FinUnion}_A$ is commutative.

(51) $\mathrm{FinUnion}_A$ is associative.

(52) $@0_A$ is a unity w.r.t. $\mathrm{FinUnion}_A$.

(53) $\mathrm{FinUnion}_A$ has a unity.

(54) $\mathbf{1}_{\mathrm{FinUnion}_A}$ is a unity w.r.t. $\mathrm{FinUnion}_A$.

(55) $\mathbf{1}_{\mathrm{FinUnion}_A} = \emptyset$.

For simplicity we adopt the following rules: X, Y are non-empty sets, A is a set, f is a function from X into $\mathrm{Fin}\, A$, and i, j, k are elements of X. The arguments of the notions defined below are the following: X which is a non-empty set; A which is a set; B which is an element of $\mathrm{Fin}\, X$; f which is a function from X into $\mathrm{Fin}\, A$. The functor $\mathrm{FinUnion}(B, f)$ yields an element of $\mathrm{Fin}\, A$ and is defined by:

$\mathrm{FinUnion}(B, f) = \mathrm{FinUnion}_A\text{-}\sum_B f$.

The following propositions are true:

(56) $\mathrm{FinUnion}(\{i\}, f) = f(i)$.

(57) $\mathrm{FinUnion}(\{i, j\}, f) = f(i) \cup f(j)$.

(58) $\mathrm{FinUnion}(\{i, j, k\}, f) = (f(i) \cup f(j)) \cup f(k)$.

(59) $\mathrm{FinUnion}(0_X, f) = \emptyset$.

(60) For every element B of $\mathrm{Fin}\, X$ holds
$\mathrm{FinUnion}(B \cup \{i\}, f) = \mathrm{FinUnion}(B, f) \cup f(i)$.

(61) For every element B of $\mathrm{Fin}\, X$ holds $\mathrm{FinUnion}(B, f) = \bigcup(f \circ B)$.

(62) For all elements B_1, B_2 of $\mathrm{Fin}\, X$ holds
$\mathrm{FinUnion}(B_1 \cup B_2, f) = \mathrm{FinUnion}(B_1, f) \cup \mathrm{FinUnion}(B_2, f)$.

(63) For every element B of $\mathrm{Fin}\, X$ for every function f from X into Y for every function g from Y into $\mathrm{Fin}\, A$ holds $\mathrm{FinUnion}(f \circ B, g) = \mathrm{FinUnion}(B, g \cdot f)$.

(64) Let A, X be non-empty sets. Let Y be a set. Let G be a binary operation on A. Suppose G is commutative and G is associative and G is idempotent. Let B be an element of $\mathrm{Fin}\, X$. Then if $B \neq \emptyset$, then for every function f from X into $\mathrm{Fin}\, Y$ for every function g from $\mathrm{Fin}\, Y$ into A such that for all elements x, y of $\mathrm{Fin}\, Y$ holds $g(x \cup y) = G(g(x), g(y))$ holds $g(\mathrm{FinUnion}(B, f)) = G\text{-}\sum_B (g \cdot f)$.

(65) Let Z be a non-empty set. Let Y be a set. Let G be a binary operation on Z. Suppose G is commutative and G is associative and G is idempotent and G has a unity. Let f be a function from X into $\mathrm{Fin}\, Y$. Let g be a function from $\mathrm{Fin}\, Y$ into Z. Then if $g(0_Y) = \mathbf{1}_G$ and for all elements x, y of $\mathrm{Fin}\, Y$ holds $g(x \cup y) = G(g(x), g(y))$, then for every element B of $\mathrm{Fin}\, X$ holds $g(\mathrm{FinUnion}(B, f)) = G\text{-}\sum_B (g \cdot f)$.

Let A be a set. The functor $\mathrm{singleton}_A$ yielding a function from A into $\mathrm{Fin}\, A$, is defined by:

for arbitrary x such that $x \in A$ holds $(\mathrm{singleton}_A)(x) = \{x\}$.

The following propositions are true:

(66) For every set A for every function f from A into $\mathrm{Fin}\, A$ holds $f = \mathrm{singleton}_A$ if and only if for arbitrary x such that $x \in A$ holds $f(x) = \{x\}$.

(67) For every non-empty set A for every function f from A into $\mathrm{Fin}\, A$ holds $f = \mathrm{singleton}_A$ if and only if for every element x of A holds $f(x) = \{x\}$.

(68) For arbitrary x for every element y of X holds $x \in \mathrm{singleton}_X(y)$ if and only if $x = y$.

(69) For all elements x, y, z of X such that $x \in \mathrm{singleton}_X(z)$ and $y \in \mathrm{singleton}_X(z)$ holds $x = y$.

(70) For every element B of $\mathrm{Fin}\, X$ for arbitrary x holds $x \in \mathrm{FinUnion}(B, f)$ if and only if there exists i being an element of X such that $i \in B$ and $x \in f(i)$.

(71) For every element B of $\mathrm{Fin}\, X$ holds $\mathrm{FinUnion}(B, \mathrm{singleton}_X) = B$.

The arguments of the notions defined below are the following: X, Y which are non-empty families of sets; g which is a function from X into Y; x which is an element of X. Then $g(x)$ is an element of Y.

Next we state a proposition

(72) Let Y, Z be sets. Let f be a function from X into $\mathrm{Fin}\, Y$. Let g be a function from $\mathrm{Fin}\, Y$ into $\mathrm{Fin}\, Z$. Then if $g(0_Y) = 0_Z$ and for all elements x,

y of Fin Y holds $g(x \cup y) = g(x) \cup g(y)$, then for every element B of Fin X holds $g(\mathrm{FinUnion}(B, f)) = \mathrm{FinUnion}(B, g \cdot f)$.

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