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Operations on Subspaces in Real Linear Space

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Summary. In this article the following operations on subspaces of real linear space are intoduced: sum, intersection and direct sum. Some theorems about those notions are proved. We define linear complement of a subspace. Some theorems about decomposition of a vector onto two subspaces and onto subspace and it's linear complement are proved. We also show that a set of subspaces with operations sum and intersection is a lattice. At the end of the article theorems that belong rather to [7], [6], [5] or [8] are proved.

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The notation and terminology used in this paper are introduced in the following papers: [1], [8], [4], [3], [6], [5], and [2]. For simplicity we adopt the following convention: V is a real linear space, W, W_1, W_2, W_3 are subspaces of V, u, u_1, u_2, v, v_1, v_2 are vectors of V, X, Y are sets, and x be arbitrary. Let us consider V, W_1, W_2. The functor $W_1 + W_2$ yielding a subspace of V, is defined by:

the vectors of $W_1 + W_2 = \{v + u : v \in W_1 \wedge u \in W_2\}$.

Let us consider V, W_1, W_2. The functor $W_1 \cap W_2$ yielding a subspace of V, is defined by:

the vectors of $W_1 \cap W_2 =$(the vectors of W_1)$\cap$(the vectors of W_2).

Next we state a number of propositions:

(1) the vectors of $W_1 + W_2 = \{v + u : v \in W_1 \wedge u \in W_2\}$.

(2) If the vectors of $W = \{v + u : v \in W_1 \wedge u \in W_2\}$, then $W = W_1 + W_2$.

(3) the vectors of $W_1 \cap W_2 =$(the vectors of W_1)$\cap$(the vectors of W_2).

(4) If the vectors of $W =$(the vectors of W_1)$\cap$(the vectors of W_2), then $W = W_1 \cap W_2$.

(5) $x \in W_1 + W_2$ if and only if there exist v_1, v_2 such that $v_1 \in W_1$ and $v_2 \in W_2$ and $x = v_1 + v_2$.

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(6) If $v \in W_1$ or $v \in W_2$, then $v \in W_1 + W_2$.

(7) $x \in W_1 \cap W_2$ if and only if $x \in W_1$ and $x \in W_2$.

(8) $W + W = W$.

(9) $W_1 + W_2 = W_2 + W_1$.

(10) $W_1 + (W_2 + W_3) = (W_1 + W_2) + W_3$.

(11) W_1 is a subspace of $W_1 + W_2$ and W_2 is a subspace of $W_1 + W_2$.

(12) W_1 is a subspace of W_2 if and only if $W_1 + W_2 = W_2$.

(13) $\mathbf{0}_V + W = W$ and $W + \mathbf{0}_V = W$.

(14) $\mathbf{0}_V + \Omega_V = V$ and $\Omega_V + \mathbf{0}_V = V$.

(15) $\Omega_V + W = V$ and $W + \Omega_V = V$.

(16) $\Omega_V + \Omega_V = V$.

(17) $W \cap W = W$.

(18) $W_1 \cap W_2 = W_2 \cap W_1$.

(19) $W_1 \cap (W_2 \cap W_3) = (W_1 \cap W_2) \cap W_3$.

(20) $W_1 \cap W_2$ is a subspace of W_1 and $W_1 \cap W_2$ is a subspace of W_2.

(21) W_1 is a subspace of W_2 if and only if $W_1 \cap W_2 = W_1$.

(22) $\mathbf{0}_V \cap W = \mathbf{0}_V$ and $W \cap \mathbf{0}_V = \mathbf{0}_V$.

(23) $\mathbf{0}_V \cap \Omega_V = \mathbf{0}_V$ and $\Omega_V \cap \mathbf{0}_V = \mathbf{0}_V$.

(24) $\Omega_V \cap W = W$ and $W \cap \Omega_V = W$.

(25) $\Omega_V \cap \Omega_V = V$.

(26) $W_1 \cap W_2$ is a subspace of $W_1 + W_2$.

(27) $W_1 \cap W_2 + W_2 = W_2$.

(28) $W_1 \cap (W_1 + W_2) = W_1$.

(29) $W_1 \cap W_2 + W_2 \cap W_3$ is a subspace of $W_2 \cap (W_1 + W_3)$.

(30) If W_1 is a subspace of W_2, then $W_2 \cap (W_1 + W_3) = W_1 \cap W_2 + W_2 \cap W_3$.

(31) $W_2 + W_1 \cap W_3$ is a subspace of $(W_1 + W_2) \cap (W_2 + W_3)$.

(32) If W_1 is a subspace of W_2, then $W_2 + W_1 \cap W_3 = (W_1 + W_2) \cap (W_2 + W_3)$.

(33) If W_1 is a subspace of W_3, then $W_1 + W_2 \cap W_3 = (W_1 + W_2) \cap W_3$.

(34) $W_1 + W_2 = W_2$ if and only if $W_1 \cap W_2 = W_1$.

(35) If W_1 is a subspace of W_2, then $W_1 + W_3$ is a subspace of $W_2 + W_3$.

(36) There exists W such that the vectors of $W =$(the vectors of W_1)$\cup$(the vectors of W_2) if and only if W_1 is a subspace of W_2 or W_2 is a subspace of W_1.

Let us consider V. The functor Subspaces V yielding a non-empty set, is defined by:

for every x holds $x \in$ Subspaces V if and only if x is a subspace of V.

In the sequel D will denote a non-empty set. We now state three propositions:

(37) If for every x holds $x \in D$ if and only if x is a subspace of V, then $D =$ Subspaces V.

(38) $x \in$ Subspaces V if and only if x is a subspace of V.

(39) $V \in \mathrm{Subspaces}\, V$.

Let us consider V, W_1, W_2. The predicate V is the direct sum of W_1 and W_2 is defined by:

$V = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_V$.

Let us consider V, W. The mode linear complement of W, which widens to the type a subspace of V, is defined by:

V is the direct sum of it and W.

One can prove the following propositions:

(40) V is the direct sum of W_1 and W_2 if and only if $V = W_1 + W_2$ and $W_1 \cap W_2 = \mathbf{0}_V$.

(41) If V is the direct sum of W_1 and W_2, then W_1 is a linear complement of W_2.

(42) If V is the direct sum of W_1 and W_2, then W_2 is a linear complement of W_1.

In the sequel L denotes a linear complement of W. One can prove the following propositions:

(43) V is the direct sum of L and W and V is the direct sum of W and L.

(44) $W + L = V$ and $L + W = V$.

(45) $W \cap L = \mathbf{0}_V$ and $L \cap W = \mathbf{0}_V$.

(46) If V is the direct sum of W_1 and W_2, then V is the direct sum of W_2 and W_1.

(47) V is the direct sum of $\mathbf{0}_V$ and Ω_V and V is the direct sum of Ω_V and $\mathbf{0}_V$.

(48) W is a linear complement of L.

(49) $\mathbf{0}_V$ is a linear complement of Ω_V and Ω_V is a linear complement of $\mathbf{0}_V$.

In the sequel C is a coset of W, C_1 is a coset of W_1, and C_2 is a coset of W_2. We now state several propositions:

(50) If $C_1 \cap C_2 \neq \emptyset$, then $C_1 \cap C_2$ is a coset of $W_1 \cap W_2$.

(51) V is the direct sum of W_1 and W_2 if and only if for every C_1, C_2 there exists v such that $C_1 \cap C_2 = \{v\}$.

(52) $W_1 + W_2 = V$ if and only if for every v there exist v_1, v_2 such that $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$.

(53) If V is the direct sum of W_1 and W_2 and $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$, then $v_1 = u_1$ and $v_2 = u_2$.

(54) Suppose $V = W_1 + W_2$ and there exists v such that for all v_1, v_2, u_1, u_2 such that $v = v_1 + v_2$ and $v = u_1 + u_2$ and $v_1 \in W_1$ and $u_1 \in W_1$ and $v_2 \in W_2$ and $u_2 \in W_2$ holds $v_1 = u_1$ and $v_2 = u_2$. Then V is the direct sum of W_1 and W_2.

In the sequel t will be an element of $[\!:$the vectors of V, the vectors of $V\!:]$. Let us consider V, t. Then $t_{\mathbf{1}}$ is a vector of V. Then $t_{\mathbf{2}}$ is a vector of V.

Let us consider V, v, W_1, W_2. Let us assume that V is the direct sum of W_1 and W_2. The functor $v \lhd (W_1, W_2)$ yields an element of $[\!:$ the vectors of V, the vectors of $V\,:\!]$ and is defined by:

$v = (v \lhd (W_1, W_2))_{\mathbf{1}} + (v \lhd (W_1, W_2))_{\mathbf{2}}$ and $(v \lhd (W_1, W_2))_{\mathbf{1}} \in W_1$ and $(v \lhd (W_1, W_2))_{\mathbf{2}} \in W_2$.

We now state a number of propositions:

(55) If V is the direct sum of W_1 and W_2 and $t_{\mathbf{1}} + t_{\mathbf{2}} = v$ and $t_{\mathbf{1}} \in W_1$ and $t_{\mathbf{2}} \in W_2$, then $t = v \lhd (W_1, W_2)$.

(56) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{1}} + (v \lhd (W_1, W_2))_{\mathbf{2}} = v$.

(57) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{1}} \in W_1$.

(58) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{2}} \in W_2$.

(59) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{1}} = (v \lhd (W_2, W_1))_{\mathbf{2}}$.

(60) If V is the direct sum of W_1 and W_2, then $(v \lhd (W_1, W_2))_{\mathbf{2}} = (v \lhd (W_2, W_1))_{\mathbf{1}}$.

(61) If $t_{\mathbf{1}} + t_{\mathbf{2}} = v$ and $t_{\mathbf{1}} \in W$ and $t_{\mathbf{2}} \in L$, then $t = v \lhd (W, L)$.

(62) $(v \lhd (W, L))_{\mathbf{1}} + (v \lhd (W, L))_{\mathbf{2}} = v$.

(63) $(v \lhd (W, L))_{\mathbf{1}} \in W$ and $(v \lhd (W, L))_{\mathbf{2}} \in L$.

(64) $(v \lhd (W, L))_{\mathbf{1}} = (v \lhd (L, W))_{\mathbf{2}}$.

(65) $(v \lhd (W, L))_{\mathbf{2}} = (v \lhd (L, W))_{\mathbf{1}}$.

In the sequel A_1, A_2 will be elements of Subspaces V. Let us consider V. The functor SubJoin V yields a binary operation on Subspaces V and is defined by:

for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubJoin}\, V)(A_1, A_2) = W_1 + W_2$.

Let us consider V. The functor SubMeet V yielding a binary operation on Subspaces V, is defined by:

for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $(\mathrm{SubMeet}\, V)(A_1, A_2) = W_1 \cap W_2$.

In the sequel o will be a binary operation on Subspaces V. The following propositions are true:

(66) If $A_1 = W_1$ and $A_2 = W_2$, then $\mathrm{SubJoin}\, V(A_1, A_2) = W_1 + W_2$.

(67) If for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $o(A_1, A_2) = W_1 + W_2$, then $o = \mathrm{SubJoin}\, V$.

(68) If $A_1 = W_1$ and $A_2 = W_2$, then $\mathrm{SubMeet}\, V(A_1, A_2) = W_1 \cap W_2$.

(69) If for all A_1, A_2, W_1, W_2 such that $A_1 = W_1$ and $A_2 = W_2$ holds $o(A_1, A_2) = W_1 \cap W_2$, then $o = \mathrm{SubMeet}\, V$.

(70) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a lattice.

(71) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a lower bound lattice.

(72) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is an upper bound lattice.

(73) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a bound lattice.

(74) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a modular lattice.

For simplicity we adopt the following convention: l will be a bound lattice, l_0 will be a lower bound lattice, l_1 will be an upper bound lattice, a, b will be elements of the carrier of l, a_0, b_0 will be elements of the carrier of l_0, and a_1, b_1 will be elements of the carrier of l_1. One can prove the following propositions:

(75) $\langle \mathrm{Subspaces}\, V, \mathrm{SubJoin}\, V, \mathrm{SubMeet}\, V \rangle$ is a complemented lattice.

(76) If W_1 is a subspace of W_2, then $W_1 \cap W_3$ is a subspace of $W_2 \cap W_3$.

(77) If $X \subseteq Y$ and $X \neq Y$, then there exists x such that $x \in Y$ and $x \notin X$.

(78) $v = v_1 + v_2$ if and only if $v_1 = v - v_2$.

(79) If for every v holds $v \in W$, then $W = V$.

(80) There exists C such that $v \in C$.

(81) $x \in v + W$ if and only if there exists u such that $u \in W$ and $x = v + u$.

(82) l is a complemented lattice if and only if for every a there exists b such that b is a complement of a.

(83) a is a complement of b if and only if $a \sqcup b = \top_l$ and $a \sqcap b = \bot_l$.

(84) If for every a_0 holds $a_0 \sqcap b_0 = b_0$, then $b_0 = \bot_{l_0}$.

(85) If for every a_1 holds $a_1 \sqcup b_1 = b_1$, then $b_1 = \top_{l_1}$.

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