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The Well Ordering Relations

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Summary. Some theorems about well ordering relations are proved. The goal of the article is to prove that every two well ordering relations are either isomorphic or one of them is isomorphic to a segment of the other. The following concepts are defined: the segment of a relation induced by an element, well founded relations, well ordering relations, the restriction of a relation to a set, and the isomorphism of two relations. A number of simple facts is presented.

The terminology and notation used here are introduced in the following papers: [2], [3], [4], [5], and [1]. For simplicity we adopt the following convention: a, b, c, x denote objects of the type Any; X, Y, Z denote objects of the type set. The scheme *Extensionality* concerns a constant $\mathcal{A}$ that has the type set, a constant $\mathcal{B}$ that has the type set and a unary predicate $\mathcal{P}$ and states that the following holds

$$\mathcal{A} = \mathcal{B}$$

provided the parameters satisfy the following conditions:

- $$\textbf{for}\ a\ \textbf{holds}\ a \in \mathcal{A}\ \textbf{iff}\ \mathcal{P}[a],$$
- $$\textbf{for}\ a\ \textbf{holds}\ a \in \mathcal{B}\ \textbf{iff}\ \mathcal{P}[a].$$

In the sequel R, S, T will have the type Relation. Let us consider R, a. The functor

$$R\ {-}\mathrm{Seg}\,a,$$

with values of the type set, is defined by

$$x \in \textbf{it}\ \textbf{iff}\ x \neq a\ \&\ \langle x, a\rangle \in R.$$

One can prove the following propositions:

(1) $\textbf{for}\ R, Y, a\ \textbf{holds}\ Y = R\ {-}\mathrm{Seg}\,(a)\ \textbf{iff}\ \textbf{for}\ b\ \textbf{holds}\ b \in Y\ \textbf{iff}\ b \neq a\ \&\ \langle b, a\rangle \in R,$

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(2) $$x \in \operatorname{field} R \ \mathbf{or}\ R -\operatorname{Seg}(x) = \emptyset.$$

We now define two new predicates. Let us consider R. The predicate

$$R \ \text{is\_well\_founded}$$

is defined by

$$\mathbf{for}\, Y \ \mathbf{st}\ Y \subseteq \operatorname{field} R \ \&\ Y \neq \emptyset \ \mathbf{ex}\, a \ \mathbf{st}\ a \in Y \ \&\ R -\operatorname{Seg}(a) \cap Y = \emptyset.$$

Let us consider X. The predicate

$$R \ \text{is\_well\_founded\_in}\ X$$

is defined by

$$\mathbf{for}\, Y \ \mathbf{st}\ Y \subseteq X \ \&\ Y \neq \emptyset \ \mathbf{ex}\, a \ \mathbf{st}\ a \in Y \ \&\ R -\operatorname{Seg}(a) \cap Y = \emptyset.$$

One can prove the following three propositions:

(3) $$\mathbf{for}\, R \ \mathbf{holds}\ R \ \text{is\_well\_founded}$$
$$\mathbf{iff}\ \mathbf{for}\, Y \ \mathbf{st}\ Y \subseteq \operatorname{field} R \ \&\ Y \neq \emptyset \ \mathbf{ex}\, a \ \mathbf{st}\ a \in Y \ \&\ R -\operatorname{Seg}(a) \cap Y = \emptyset,$$

(4) $$\mathbf{for}\, R{,}X \ \mathbf{holds}\ R \ \text{is\_well\_founded\_in}\ X$$
$$\mathbf{iff}\ \mathbf{for}\, Y \ \mathbf{st}\ Y \subseteq X \ \&\ Y \neq \emptyset \ \mathbf{ex}\, a \ \mathbf{st}\ a \in Y \ \&\ R -\operatorname{Seg}(a) \cap Y = \emptyset,$$

(5) $$R \ \text{is\_well\_founded}\ \mathbf{iff}\ R \ \text{is\_well\_founded\_in}\ \operatorname{field} R.$$

We now define two new predicates. Let us consider R. The predicate

$$R \ \text{is\_well-ordering-relation}$$

is defined by

$$R \ \text{is\_reflexive}$$
$$\&\ R \ \text{is\_transitive}\ \&\ R \ \text{is\_antisymmetric}\ \&\ R \ \text{is\_connected}\ \&\ R \ \text{is\_well\_founded}\,.$$

Let us consider X. The predicate

$$R \ \text{well\_orders}\ X$$

is defined by

$$R \ \text{is\_reflexive\_in}\ X \ \&\ R \ \text{is\_transitive\_in}\ X$$
$$\&\ R \ \text{is\_antisymmetric\_in}\ X \ \&\ R \ \text{is\_connected\_in}\ X \ \&\ R \ \text{is\_well\_founded\_in}\ X.$$

The following propositions are true:

(6) $$\mathbf{for}\, R \ \mathbf{holds}\ R \ \text{is\_well-ordering-relation}\ \mathbf{iff}\ R \ \text{is\_reflexive}$$
$$\&\ R \ \text{is\_transitive}\ \&\ R \ \text{is\_antisymmetric}\ \&\ R \ \text{is\_connected}\ \&\ R \ \text{is\_well\_founded}\,,$$

(7) $\textbf{for}\ R{,}X\ \textbf{holds}\ R\ \text{well\_orders}\ X\ \textbf{iff}\ R\ \text{is\_reflexive\_in}\ X\ \&\ R\ \text{is\_transitive\_in}\ X\ \&\ R\ \text{is\_antisymmetric\_in}\ X\ \&\ R\ \text{is\_connected\_in}\ X\ \&\ R\ \text{is\_well\_founded\_in}\ X,$

$$R\ \text{well\_orders field}\ R\ \textbf{iff}\ R\,\text{is\_well-ordering-relation}, \tag{8}$$

$$R\ \text{well\_orders}\ X\ \textbf{implies}\ \textbf{for}\ Y\ \textbf{st}\ Y \subseteq X\ \&\ Y \neq \emptyset\ \textbf{ex}\ a\ \textbf{st}\ a \in Y\ \&\ \textbf{for}\ b\ \textbf{st}\ b \in Y\ \textbf{holds}\ \langle a,b\rangle \in R, \tag{9}$$

$$R\,\text{is\_well-ordering-relation}\ \textbf{implies}\ \textbf{for}\ Y\ \textbf{st}\ Y \subseteq \text{field}\,R\ \&\ Y \neq \emptyset\ \textbf{ex}\ a\ \textbf{st}\ a \in Y\ \&\ \textbf{for}\ b\ \textbf{st}\ b \in Y\ \textbf{holds}\ \langle a,b\rangle \in R, \tag{10}$$

$$\textbf{for}\ R\ \textbf{st}\ R\,\text{is\_well-ordering-relation}\ \&\ \text{field}\,R \neq \emptyset\ \textbf{ex}\ a\ \textbf{st}\ a \in \text{field}\,R\ \&\ \textbf{for}\ b\ \textbf{st}\ b \in \text{field}\,R\ \textbf{holds}\ \langle a,b\rangle \in R, \tag{11}$$

(12) $\textbf{for}\ R\ \textbf{st}\ R\,\text{is\_well-ordering-relation}\ \&\ \text{field}\,R \neq \emptyset\ \textbf{for}\ a\ \textbf{st}\ a \in \text{field}\,R\ \textbf{holds}\ (\textbf{for}\ b\ \textbf{st}\ b \in \text{field}\,R\ \textbf{holds}\ \langle b,a\rangle \in R)\ \textbf{or}\ \textbf{ex}\ b\ \textbf{st}\ b \in \text{field}\,R\ \&\ \langle a,b\rangle \in R\ \&\ \textbf{for}\ c\ \textbf{st}\ c \in \text{field}\,R\ \&\ \langle a,c\rangle \in R\ \textbf{holds}\ c = a\ \textbf{or}\ \langle b,c\rangle \in R.$

In the sequel F, G have the type Function. Next we state a proposition

$$R\ -\text{Seg}\,(a) \subseteq \text{field}\,R. \tag{13}$$

Let us consider R, Y. The functor

$$R\,|^2\,Y,$$

yields the type Relation and is defined by

$$\textbf{it} = R \cap [\![Y, Y]\!].$$

We now state a number of propositions:

$$R\,|^2\,Y = R \cap [\![Y, Y]\!], \tag{14}$$

$$R\,|^2\,X \subseteq R\ \&\ R\,|^2\,X \subseteq [\![X, X]\!], \tag{15}$$

$$x \in R\,|^2\,X\ \textbf{iff}\ x \in R\ \&\ x \in [\![X, X]\!], \tag{16}$$

$$R\,|^2\,X = X \mid R \mid X, \tag{17}$$

$$R\,|^2\,X = X \mid (R \mid X), \tag{18}$$

$$x \in \text{field}\,(R\,|^2\,X)\ \textbf{implies}\ x \in \text{field}\,R\ \&\ x \in X, \tag{19}$$

$$\text{field}\,(R\,|^2\,X) \subseteq \text{field}\,R\ \&\ \text{field}\,(R\,|^2\,X) \subseteq X, \tag{20}$$

$$(R\,|^2\,X)\ -\text{Seg}\,(a) \subseteq R\ -\text{Seg}\,(a), \tag{21}$$

$$R \text{ is\_reflexive } \textbf{implies } R \,|^2\, X \text{ is\_reflexive}, \tag{22}$$

$$R \text{ is\_connected } \textbf{implies } R \,|^2\, Y \text{ is\_connected}, \tag{23}$$

$$R \text{ is\_transitive } \textbf{implies } R \,|^2\, Y \text{ is\_transitive}, \tag{24}$$

$$R \text{ is\_antisymmetric } \textbf{implies } R \,|^2\, Y \text{ is\_antisymmetric}, \tag{25}$$

$$(R \,|^2\, X) \,|^2\, Y = R \,|^2\, (X \cap Y), \tag{26}$$

$$(R \,|^2\, X) \,|^2\, Y = (R \,|^2\, Y) \,|^2\, X, \tag{27}$$

$$(R \,|^2\, Y) \,|^2\, Y = R \,|^2\, Y, \tag{28}$$

$$Z \subseteq Y \textbf{ implies } (R \,|^2\, Y) \,|^2\, Z = R \,|^2\, Z, \tag{29}$$

$$R \,|^2\, \operatorname{field} R = R, \tag{30}$$

$$R \text{ is\_well\_founded } \textbf{implies } R \,|^2\, X \text{ is\_well\_founded}, \tag{31}$$

$$R \text{ is\_well-ordering-relation } \textbf{implies } R \,|^2\, Y \text{ is\_well-ordering-relation}, \tag{32}$$

$$\begin{gathered} R \text{ is\_well-ordering-relation} \\ \textbf{implies } R - \operatorname{Seg}(a) \subseteq R - \operatorname{Seg}(b) \textbf{ or } R - \operatorname{Seg}(b) \subseteq R - \operatorname{Seg}(a), \end{gathered} \tag{33}$$

$$R \text{ is\_well-ordering-relation } \textbf{implies } R \,|^2\, (R - \operatorname{Seg}(a)) \text{ is\_well-ordering-relation}, \tag{34}$$

$$\begin{gathered} R \text{ is\_well-ordering-relation} \;\&\; a \in \operatorname{field} R \;\&\; b \in R - \operatorname{Seg}(a) \\ \textbf{implies } (R \,|^2\, (R - \operatorname{Seg}(a))) - \operatorname{Seg}(b) = R - \operatorname{Seg}(b), \end{gathered} \tag{35}$$

$$\begin{gathered} R \text{ is\_well-ordering-relation} \;\&\; Y \subseteq \operatorname{field} R \textbf{ implies} \\ (Y = \operatorname{field} R \textbf{ or } (\textbf{ex}\, a \textbf{ st } a \in \operatorname{field} R \;\&\; Y = R - \operatorname{Seg}(a)) \\ \textbf{iff for } a \textbf{ st } a \in Y \textbf{ for } b \textbf{ st } \langle b, a \rangle \in R \textbf{ holds } b \in Y), \end{gathered} \tag{36}$$

$$\begin{gathered} R \text{ is\_well-ordering-relation} \;\&\; a \in \operatorname{field} R \;\&\; b \in \operatorname{field} R \\ \textbf{implies } (\langle a, b \rangle \in R \textbf{ iff } R - \operatorname{Seg}(a) \subseteq R - \operatorname{Seg}(b)), \end{gathered} \tag{37}$$

$$\begin{gathered} R \text{ is\_well-ordering-relation} \;\&\; a \in \operatorname{field} R \;\&\; b \in \operatorname{field} R \\ \textbf{implies } (R - \operatorname{Seg}(a) \subseteq R - \operatorname{Seg}(b) \textbf{ iff } a = b \textbf{ or } a \in R - \operatorname{Seg}(b)), \end{gathered} \tag{38}$$

$$R \text{ is\_well-ordering-relation} \;\&\; X \subseteq \operatorname{field} R \textbf{ implies } \operatorname{field}(R \,|^2\, X) = X, \tag{39}$$

$$R \text{ is\_well-ordering-relation } \textbf{implies } \operatorname{field}(R \,|^2\, R - \operatorname{Seg}(a)) = R - \operatorname{Seg}(a), \tag{40}$$

$$\begin{gathered} R \text{ is\_well-ordering-relation } \textbf{implies} \\ \textbf{for } Z \textbf{ st for } a \textbf{ st } a \in \operatorname{field} R \;\&\; R - \operatorname{Seg}(a) \subseteq Z \textbf{ holds } a \in Z \textbf{ holds } \operatorname{field} R \subseteq Z, \end{gathered} \tag{41}$$

(42)
$$R\,\mathrm{is\_well\text{-}ordering\text{-}relation}\ \&\ a \in \mathrm{field}\,R\ \&\ b \in \mathrm{field}\,R\ \&\ (\mathbf{for}\,c\ \mathbf{st}\ c \in R\ {-}\mathrm{Seg}\,(a)\ \mathbf{holds}\ \langle c, b\rangle \in R\ \&\ c \neq b)\ \mathbf{implies}\ \langle a, b\rangle \in R,$$

(43)
$$R\,\mathrm{is\_well\text{-}ordering\text{-}relation}\ \&\ \mathrm{dom}\,F = \mathrm{field}\,R\ \&\ \mathrm{rng}\,F \subseteq \mathrm{field}\,R\ \&\ (\mathbf{for}\,a{,}b\ \mathbf{st}\ \langle a, b\rangle \in R\ \&\ a \neq b\ \mathbf{holds}\ \langle F\,.a{,}F\,.b\rangle \in R\ \&\ F\,.a \neq F\,.b)\ \mathbf{implies\ for}\,a\ \mathbf{st}\ a \in \mathrm{field}\,R\ \mathbf{holds}\ \langle a, F\,.a\rangle \in R.$$

Let us consider R, S, F. The predicate

$$F\ \mathrm{is\_isomorphism\_of}\ R, S$$

is defined by

$$\mathrm{dom}\,F = \mathrm{field}\,R\ \&\ \mathrm{rng}\,F = \mathrm{field}\,S\ \&\ F\,\mathrm{is\_one\text{-}to\text{-}one}\ \&\ \mathbf{for}\,a{,}b\ \mathbf{holds}\ \langle a, b\rangle \in R\ \mathbf{iff}\ a \in \mathrm{field}\,R\ \&\ b \in \mathrm{field}\,R\ \&\ \langle F\,.a{,}F\,.b\rangle \in S.$$

Next we state two propositions:

(44)
$$F\ \mathrm{is\_isomorphism\_of}\ R, S\ \mathbf{iff}\ \mathrm{dom}\,F = \mathrm{field}\,R\ \&\ \mathrm{rng}\,F = \mathrm{field}\,S\ \&\ F\,\mathrm{is\_one\text{-}to\text{-}one}\ \&\ \mathbf{for}\,a{,}b\ \mathbf{holds}\ \langle a, b\rangle \in R\ \mathbf{iff}\ a \in \mathrm{field}\,R\ \&\ b \in \mathrm{field}\,R\ \&\ \langle F\,.a{,}F\,.b\rangle \in S,$$

(45)
$$F\ \mathrm{is\_isomorphism\_of}\ R, S\ \mathbf{implies\ for}\,a{,}b\ \mathbf{st}\ \langle a, b\rangle \in R\ \&\ a \neq b\ \mathbf{holds}\ \langle F\,.a{,}F\,.b\rangle \in S\ \&\ F\,.a \neq F\,.b.$$

Let us consider R, S. The predicate

$R, S\,\mathrm{are\_isomorphic}$ is defined by $\mathbf{ex}\,F\ \mathbf{st}\ F\ \mathrm{is\_isomorphism\_of}\ R, S.$

We now state a number of propositions:

(46)
$$R, S\,\mathrm{are\_isomorphic}\ \mathbf{iff}\ \mathbf{ex}\,F\ \mathbf{st}\ F\ \mathrm{is\_isomorphism\_of}\ R, S,$$

(47)
$$\mathrm{id}\,(\mathrm{field}\,R)\ \mathrm{is\_isomorphism\_of}\ R, R,$$

(48)
$$R, R\,\mathrm{are\_isomorphic}\,,$$

(49)
$$F\ \mathrm{is\_isomorphism\_of}\ R, S\ \mathbf{implies}\ F^{\,\text{-}1}\ \mathrm{is\_isomorphism\_of}\ S, R,$$

(50)
$$R, S\,\mathrm{are\_isomorphic}\ \mathbf{implies}\ S, R\,\mathrm{are\_isomorphic}\,,$$

(51)
$$F\ \mathrm{is\_isomorphism\_of}\ R, S\ \&\ G\ \mathrm{is\_isomorphism\_of}\ S, T\ \mathbf{implies}\ G \cdot F\ \mathrm{is\_isomorphism\_of}\ R, T,$$

(52)
$$R, S\,\mathrm{are\_isomorphic}\ \&\ S, T\,\mathrm{are\_isomorphic}\ \mathbf{implies}\ R, T\,\mathrm{are\_isomorphic}\,,$$

$$F \text{ is\_isomorphism\_of } R,S \textbf{ implies } (R \text{ is\_reflexive} \textbf{ implies } S \text{ is\_reflexive}) \mathrel{\&} (R \text{ is\_transitive} \textbf{ implies } S \text{ is\_transitive}) \mathrel{\&} (R \text{ is\_connected} \textbf{ implies } S \text{ is\_connected}) \mathrel{\&} (R \text{ is\_antisymmetric} \textbf{ implies } S \text{ is\_antisymmetric}) \mathrel{\&} (R \text{ is\_well\_founded} \textbf{ implies } S \text{ is\_well\_founded}), \tag{53}$$

$$R \text{ is\_well-ordering-relation} \mathrel{\&} F \text{ is\_isomorphism\_of } R,S \textbf{ implies } S \text{ is\_well-ordering-relation}, \tag{54}$$

$$R \text{ is\_well-ordering-relation} \textbf{ implies for } F,G \textbf{ st } F \text{ is\_isomorphism\_of } R,S \mathrel{\&} G \text{ is\_isomorphism\_of } R,S \textbf{ holds } F = G. \tag{55}$$

Let us consider R, S. Assume that the following holds

$$R \text{ is\_well-ordering-relation} \mathrel{\&} R,S \text{ are\_isomorphic}.$$

The functor

$$\text{canonical\_isomorphism\_of}\,(R,S),$$

yields the type Function and is defined by

$$\textbf{it} \text{ is\_isomorphism\_of } R,S.$$

The following propositions are true:

$$R \text{ is\_well-ordering-relation} \mathrel{\&} R,S \text{ are\_isomorphic} \textbf{ implies } (F = \text{canonical\_isomorphism\_of}\,(R,S) \textbf{ iff } F \text{ is\_isomorphism\_of } R,S), \tag{56}$$

$$R \text{ is\_well-ordering-relation} \textbf{ implies for } a \textbf{ st } a \in \text{field}\, R \textbf{ holds not } R, R\,|^2\,(R\,-\text{Seg}\,(a)) \text{ are\_isomorphic}, \tag{57}$$

$$R \text{ is\_well-ordering-relation} \mathrel{\&} a \in \text{field}\, R \mathrel{\&} b \in \text{field}\, R \mathrel{\&} a \neq b \textbf{ implies not } R\,|^2\,(R\,-\text{Seg}\,(a)), R\,|^2\,(R\,-\text{Seg}\,(b)) \text{ are\_isomorphic}, \tag{58}$$

$$R \text{ is\_well-ordering-relation} \mathrel{\&} Z \subseteq \text{field}\, R \mathrel{\&} F \text{ is\_isomorphism\_of } R,S \textbf{ implies } F \mid Z \text{ is\_isomorphism\_of } R\,|^2\,Z, S\,|^2\,(F\,^\circ\,Z) \mathrel{\&} R\,|^2\,Z, S\,|^2\,(F\,^\circ\,Z) \text{ are\_isomorphic}, \tag{59}$$

$$R \text{ is\_well-ordering-relation} \mathrel{\&} F \text{ is\_isomorphism\_of } R,S \textbf{ implies for } a \textbf{ st } a \in \text{field}\, R \textbf{ ex } b \textbf{ st } b \in \text{field}\, S \mathrel{\&} F\,^\circ\,(R\,-\text{Seg}\,(a)) = S\,-\text{Seg}\,(b), \tag{60}$$

$$R \text{ is\_well-ordering-relation} \mathrel{\&} F \text{ is\_isomorphism\_of } R,S \textbf{ implies for } a \textbf{ st } a \in \text{field}\, R \textbf{ ex } b \textbf{ st } b \in \text{field}\, S \mathrel{\&} R\,|^2\,(R\,-\text{Seg}\,(a)), S\,|^2\,(S\,-\text{Seg}\,(b)) \text{ are\_isomorphic}, \tag{61}$$

(62) $$R \text{ is\_well-ordering-relation } \& \ S \text{ is\_well-ordering-relation } \& \ a \in \text{field } R \ \& \\ b \in \text{field } S \ \& \ c \in \text{field } S \ \& \ R, S\,|^2\,(S\,-\text{Seg}\,(b)) \text{ are\_isomorphic} \\ \& \ R\,|^2\,(R\,-\text{Seg}\,(a)), S\,|^2\,(S\,-\text{Seg}\,(c)) \text{ are\_isomorphic} \\ \textbf{implies } S\,-\text{Seg}\,(c) \subseteq S\,-\text{Seg}\,(b) \ \& \ \langle c, b\rangle \in S,$$

(63) $$R \text{ is\_well-ordering-relation } \& \ S \text{ is\_well-ordering-relation } \textbf{implies} \\ R, S \text{ are\_isomorphic } \textbf{or} \\ (\textbf{ex}\, a \textbf{ st } a \in \text{field } R \ \& \ R\,|^2\,(R\,-\text{Seg}\,(a)), S \text{ are\_isomorphic}) \\ \textbf{or ex}\, a \textbf{ st } a \in \text{field } S \ \& \ R, S\,|^2\,(S\,-\text{Seg}\,(a)) \text{ are\_isomorphic}\,,$$

(64) $$Y \subseteq \text{field } R \ \& \ R \text{ is\_well-ordering-relation } \textbf{implies } R, R\,|^2\,Y \text{ are\_isomorphic} \\ \textbf{or ex}\, a \textbf{ st } a \in \text{field } R \ \& \ R\,|^2\,(R\,-\text{Seg}\,(a)), R\,|^2\,Y \text{ are\_isomorphic}\,.$$

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