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Introduction to Lattice Theory

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Summary. A lattice is defined as an algebra on a nonempty set with binary operations join and meet which are commutative and associative, and satisfy the absorption identities. The following kinds of lattices are considered: distributive, modular, bounded (with zero and unit elements), complemented, and Boolean (with complement). The article includes also theorems which immediately follow from definitions.

The terminology and notation used in this paper are introduced in the papers [1] and [2]. The scheme *BooleDomBinOpLam* deals with a constant $\mathcal{A}$ that has the type BOOLE_DOMAIN and a binary functor $\mathcal{F}$ yielding values of the type Element **of** $\mathcal{A}$ and states that the following holds

$$\textbf{ex}\, o \,\textbf{being}\ \text{Binary\_Operation}\ \textbf{of}\ \mathcal{A}$$
$$\textbf{st for}\ a{,}b\ \textbf{being}\ \text{Element}\ \textbf{of}\ \mathcal{A}\ \textbf{holds}\ o\,.(a,b) = \mathcal{F}(a,b)$$

for all values of the parameters.

We consider structures LattStr, which are systems

$$\langle\!\langle \text{carrier}\,,\text{join}\,,\text{meet}\rangle\!\rangle$$

where carrier has the type DOMAIN, and join, meet have the type Binary_Operation **of the** carrier. In the sequel G has the type LattStr; p, q, r have the type Element **of the** carrier **of** G. We now define two new functors. Let us consider G, p, q. The functor

$$p \sqcup q,$$

yields the type Element **of the** carrier **of** G and is defined by

$$\textbf{it} = (\textbf{the}\ \text{join}\ \textbf{of}\ G)\,.(p,q).$$

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The functor

$$p \sqcap q,$$

with values of the type Element **of the** carrier **of** G, is defined by

$$\textbf{it} = (\textbf{the}\,\text{meet}\ \textbf{of}\ G).(p, q).$$

The following propositions are true:

$$p \sqcup q = (\textbf{the}\,\text{join}\ \textbf{of}\ G).(p, q), \tag{1}$$

$$p \sqcap q = (\textbf{the}\,\text{meet}\ \textbf{of}\ G).(p, q). \tag{2}$$

Let us consider G, p, q. The predicate

$$p \sqsubseteq q \qquad \text{is defined by} \qquad p \sqcup q = q.$$

We now state a proposition

$$p \sqsubseteq q\ \textbf{iff}\ p \sqcup q = q. \tag{3}$$

The mode

$$\text{Lattice}\,,$$

which widens to the type LattStr, is defined by

$$(\textbf{for}\ a,b\ \textbf{being}\ \text{Element}\ \textbf{of the}\ \text{carrier}\ \textbf{of it holds}\ a \sqcup b = b \sqcup a)\ \&$$
$$(\textbf{for}\ a,b,c\ \textbf{being}\ \text{Element}\ \textbf{of the}\ \text{carrier}\ \textbf{of it holds}\ a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c)\ \&$$
$$(\textbf{for}\ a,b\ \textbf{being}\ \text{Element}\ \textbf{of the}\ \text{carrier}\ \textbf{of it holds}\ (a \sqcap b) \sqcup b = b)\ \&$$
$$(\textbf{for}\ a,b\ \textbf{being}\ \text{Element}\ \textbf{of the}\ \text{carrier}\ \textbf{of it holds}\ a \sqcap b = b \sqcap a)\ \&$$
$$(\textbf{for}\ a,b,c\ \textbf{being}\ \text{Element}\ \textbf{of the}\ \text{carrier}\ \textbf{of it holds}\ a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c)$$
$$\&\ \textbf{for}\ a,b\ \textbf{being}\ \text{Element}\ \textbf{of the}\ \text{carrier}\ \textbf{of it holds}\ a \sqcap (a \sqcup b) = a.$$

One can prove the following proposition

$$\begin{gathered}(\textbf{for}\ p,q\ \textbf{holds}\ p \sqcup q = q \sqcup p)\ \&\ (\textbf{for}\ p,q,r\ \textbf{holds}\ p \sqcup (q \sqcup r) = (p \sqcup q) \sqcup r)\ \&\\ (\textbf{for}\ p,q\ \textbf{holds}\ (p \sqcap q) \sqcup q = q)\ \&\ (\textbf{for}\ p,q\ \textbf{holds}\ p \sqcap q = q \sqcap p)\\ \&\ (\textbf{for}\ p,q,r\ \textbf{holds}\ p \sqcap (q \sqcap r) = (p \sqcap q) \sqcap r)\ \&\ (\textbf{for}\ p,q\ \textbf{holds}\ p \sqcap (p \sqcup q) = p)\\ \textbf{implies}\ G\ \textbf{is}\ \text{Lattice}\,.\end{gathered} \tag{4}$$

In the sequel L has the type Lattice; a, b, c have the type Element **of the** carrier **of** L. One can prove the following propositions:

$$a \sqcup b = b \sqcup a, \tag{5}$$

$$a \sqcap b = b \sqcap a, \tag{6}$$

$$a \sqcup (b \sqcup c) = (a \sqcup b) \sqcup c, \tag{7}$$

(8) $$a \sqcap (b \sqcap c) = (a \sqcap b) \sqcap c,$$

(9) $$(a \sqcap b) \sqcup b = b \;\&\; b \sqcup (a \sqcap b) = b \;\&\; b \sqcup (b \sqcap a) = b \;\&\; (b \sqcap a) \sqcup b = b,$$

(10) $$a \sqcap (a \sqcup b) = a \;\&\; (a \sqcup b) \sqcap a = a \;\&\; (b \sqcup a) \sqcap a = a \;\&\; a \sqcap (b \sqcup a) = a.$$

The mode

Distributive_Lattice ,

which widens to the type Lattice, is defined by

for a,b,c **being** Element **of the** carrier **of it holds** $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$.

Next we state a proposition

(11) (**for** a,b,c **holds** $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$)
implies L **is** Distributive_Lattice .

The mode

Modular_Lattice ,

which widens to the type Lattice, is defined by

for a,b,c **being** Element **of the** carrier **of it st** $a \sqsubseteq c$ **holds** $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap c$.

One can prove the following proposition

(12) (**for** a,b,c **st** $a \sqsubseteq c$ **holds** $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap c$)
implies L **is** Modular_Lattice .

The mode

Lower_Bound_Lattice ,

which widens to the type Lattice, is defined by

ex c **being** Element **of the** carrier **of it**
st for a **being** Element **of the** carrier **of it holds** $c \sqcap a = c$.

Next we state a proposition

(13) (**ex** c **st for** a **holds** $c \sqcap a = c$) **implies** L **is** Lower_Bound_Lattice .

The mode

Upper_Bound_Lattice ,

which widens to the type Lattice, is defined by

ex c **being** Element **of the** carrier **of it**
st for a **being** Element **of the** carrier **of it holds** $c \sqcup a = c$.

One can prove the following proposition

(14) (**ex** c **st for** a **holds** $c \sqcup a = c$) **implies** L **is** Upper_Bound_Lattice .

The mode

Bound_Lattice ,

which widens to the type Lattice, is defined by

it is Lower_Bound_Lattice & **it is** Upper_Bound_Lattice .

Next we state a proposition

(15) L **is** Lower_Bound_Lattice & L **is** Upper_Bound_Lattice **implies** L **is** Bound_Lattice .

Let us consider L. Assume that the following holds

ex c **st for** a **holds** $c \sqcap a = c$.

The functor

$\perp L$,

yields the type Element **of the** carrier **of** L and is defined by

it $\sqcap a =$ **it** .

Let L have the type Lower_Bound_Lattice. Let us note that it makes sense to consider the following functor on a restricted area. Then

$\perp L$ is Element **of the** carrier **of** L.

Let us consider L. Assume that the following holds

ex c **st for** a **holds** $c \sqcup a = c$.

The functor

$\top L$,

with values of the type Element **of the** carrier **of** L, is defined by

it $\sqcup a =$ **it** .

Let L have the type Upper_Bound_Lattice. Let us note that it makes sense to consider the following functor on a restricted area. Then

$\top L$ is Element **of the** carrier **of** L.

Let L have the type Bound_Lattice. Let us note that it makes sense to consider the following functors on restricted areas. Then

$\perp L$ is Element **of the** carrier **of** L,

$\top L$ is Element **of the** carrier **of** L.

Let us consider L, a, b. Assume that the following holds

L **is** Bound_Lattice .

The predicate

a is_a_complement_of b is defined by $a \sqcup b = \top L$ & $a \sqcap b = \perp L$.

The mode

Lattice_with_Complement ,

which widens to the type Bound_Lattice, is defined by

for b **being** Element **of the** carrier **of it**
ex a **being** Element **of the** carrier **of it st** a is_a_complement_of b.

The mode

Boolean_Lattice ,

which widens to the type Lattice_with_Complement, is defined by

it is Distributive_Lattice .

The following propositions are true:

(16) $a \sqcup b = b$ **iff** $a \sqcap b = a$,

(17) $a \sqcup a = a$,

(18) $a \sqcap a = a$,

(19) **for** L **holds** (**for** a,b,c **holds** $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$)
iff for a,b,c **holds** $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$,

(20) $a \sqsubseteq b$ **iff** $a \sqcup b = b$,

(21) $a \sqsubseteq b$ **iff** $a \sqcap b = a$,

(22) $a \sqsubseteq a \sqcup b$,

(23) $a \sqcap b \sqsubseteq a$,

(24) $a \sqsubseteq a$,

(25) $a \sqsubseteq b$ & $b \sqsubseteq c$ **implies** $a \sqsubseteq c$,

(26) $a \sqsubseteq b$ & $b \sqsubseteq a$ **implies** $a = b$,

(27) $a \sqsubseteq b$ **implies** $a \sqcap c \sqsubseteq b \sqcap c$,

(28) $$a \sqsubseteq b \textbf{ implies } c \sqcap a \sqsubseteq c \sqcap b,$$

(29) $$(\textbf{for } a,b,c \textbf{ holds } (a \sqcap b) \sqcup (b \sqcap c) \sqcup (c \sqcap a) = (a \sqcup b) \sqcap (b \sqcup c) \sqcap (c \sqcup a)) \textbf{ implies } L \textbf{ is } \text{Distributive\_Lattice}.$$

In the sequel L denotes an object of the type Distributive_Lattice; a, b, c denote objects of the type Element **of the** carrier **of** L. One can prove the following propositions:

(30) $$\textbf{for } L \textbf{ holds } (\textbf{for } a,b,c \textbf{ holds } a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)) \;\&\; \textbf{for } a,b,c \textbf{ holds } (b \sqcup c) \sqcap a = (b \sqcap a) \sqcup (c \sqcap a),$$

(31) $$\textbf{for } L \textbf{ holds } (\textbf{for } a,b,c \textbf{ holds } a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)) \;\&\; \textbf{for } a,b,c \textbf{ holds } (b \sqcap c) \sqcup a = (b \sqcup a) \sqcap (c \sqcup a),$$

(32) $$c \sqcap a = c \sqcap b \;\&\; c \sqcup a = c \sqcup b \textbf{ implies } a = b,$$

(33) $$a \sqcap c = b \sqcap c \;\&\; a \sqcup c = b \sqcup c \textbf{ implies } a = b,$$

(34) $$(a \sqcup b) \sqcap (b \sqcup c) \sqcap (c \sqcup a) = (a \sqcap b) \sqcup (b \sqcap c) \sqcup (c \sqcap a),$$

(35) $$L \textbf{ is } \text{Modular\_Lattice}.$$

In the sequel L has the type Modular_Lattice; a, b, c have the type Element **of the** carrier **of** L. One can prove the following two propositions:

(36) $$a \sqsubseteq c \textbf{ implies } a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap c,$$

(37) $$c \sqsubseteq a \textbf{ implies } a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup c.$$

In the sequel L has the type Lower_Bound_Lattice; a, c have the type Element **of the** carrier **of** L. We now state four propositions:

(38) $$\textbf{ex } c \textbf{ st for } a \textbf{ holds } c \sqcap a = c,$$

(39) $$\bot L \sqcup a = a \;\&\; a \sqcup \bot L = a,$$

(40) $$\bot L \sqcap a = \bot L \;\&\; a \sqcap \bot L = \bot L,$$

(41) $$\bot L \sqsubseteq a.$$

In the sequel L denotes an object of the type Upper_Bound_Lattice; a, c denote objects of the type Element **of the** carrier **of** L. The following four propositions are true:

(42) $$\textbf{ex } c \textbf{ st for } a \textbf{ holds } c \sqcup a = c,$$

(43) $$\top L \sqcap a = a \;\&\; a \sqcap \top L = a,$$

$$\top L \sqcup a = \top L \;\&\; a \sqcup \top L = \top L, \tag{44}$$

$$a \sqsubseteq \top L. \tag{45}$$

In the sequel L has the type Lattice_with_Complement; a, b have the type Element **of the** carrier **of** L. One can prove the following proposition

$$\textbf{ex}\, a \;\textbf{st}\; a \text{ is\_a\_complement\_of } b. \tag{46}$$

In the sequel L has the type Lattice. The arguments of the notions defined below are the following: L which is an object of the type reserved above; x which is an object of the type Element **of the** carrier **of** L. Assume that the following holds

$$L \;\textbf{is}\; \text{Boolean\_Lattice}.$$

The functor

$$x^{\mathrm{c}},$$

yields the type Element **of the** carrier **of** L and is defined by

$$\textbf{it} \text{ is\_a\_complement\_of } x.$$

The arguments of the notions defined below are the following: L which is an object of the type Boolean_Lattice; x which is an object of the type Element **of the** carrier **of** L. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$x^{\mathrm{c}} \quad \text{is} \quad \text{Element } \textbf{of the} \text{ carrier } \textbf{of } L.$$

In the sequel L will denote an object of the type Boolean_Lattice; a, b will denote objects of the type Element **of the** carrier **of** L. We now state several propositions:

$$a^{\mathrm{c}} \sqcap a = \bot L \;\&\; a \sqcap a^{\mathrm{c}} = \bot L, \tag{47}$$

$$a^{\mathrm{c}} \sqcup a = \top L \;\&\; a \sqcup a^{\mathrm{c}} = \top L, \tag{48}$$

$$a^{\mathrm{c}\,\mathrm{c}} = a, \tag{49}$$

$$(a \sqcap b)^{\mathrm{c}} = a^{\mathrm{c}} \sqcup b^{\mathrm{c}}, \tag{50}$$

$$(a \sqcup b)^{\mathrm{c}} = a^{\mathrm{c}} \sqcap b^{\mathrm{c}}, \tag{51}$$

$$b \sqcap a = \bot L \;\textbf{iff}\; b \sqsubseteq a^{\mathrm{c}}, \tag{52}$$

$$a \sqsubseteq b \;\textbf{implies}\; b^{\mathrm{c}} \sqsubseteq a^{\mathrm{c}}. \tag{53}$$

In the sequel L will have the type Bound_Lattice; a, b will have the type Element **of the** carrier **of** L. We now state three propositions:

$$L \;\textbf{is}\; \text{Lower\_Bound\_Lattice} \;\&\; L \;\textbf{is}\; \text{Upper\_Bound\_Lattice}, \tag{54}$$

(55) a is_a_complement_of b **iff** $a \sqcup b = \top L$ & $a \sqcap b = \perp L$,

(56) (**for** b **ex** a **st** a is_a_complement_of b) **implies** L **is** Lattice_with_Complement .

In the sequel L has the type Lattice_with_Complement. One can prove the following proposition

(57) L **is** Distributive_Lattice **implies** L **is** Boolean_Lattice .

In the sequel L has the type Boolean_Lattice. The following two propositions are true:

(58) L **is** Lattice_with_Complement ,

(59) L **is** Distributive_Lattice .

References

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