Segments of Natural Numbers and Finite Sequences

Grzegorz Bancerek¹ Warsaw University Białystok Krzysztof Hryniewiecki² Warsaw University Warsaw

Summary. We define the notion of an initial segment of natural numbers and prove a number of their properties. Using this notion we introduce finite sequences, subsequences, the empty sequence, a sequence of a domain, and the operation of concatenation of two sequences.

The papers [4], [5], [2], [3], and [1] provide the notation and terminology for this paper. For simplicity we adopt the following convention: k, l, m, n, k1, k2 denote objects of the type Nat; X denotes an object of the type set; x, y, z, y1, y2 denote objects of the type Any; f denotes an object of the type Function. Let us consider n. The functor

 $\operatorname{Seg} n$,

with values of the type set, is defined by

$$\mathbf{it} = \{ k : 1 \le k \& k \le n \}.$$

Let us consider n. Let us note that it makes sense to consider the following functor on a restricted area. Then

 $\operatorname{Seg} n$ is **set of** Nat.

One can prove the following propositions:

(1)
$$\operatorname{Seg} n = \{k : 1 \le k \& k \le n\},\$$

(2) $x \in \operatorname{Seg} n \text{ implies } x \text{ is Nat},$

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(3) k \in \operatorname{Seg} n \text{ iff } 1 \le k \& k \le n,
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(4) $\operatorname{Seg} 0 = \emptyset \& \operatorname{Seg} 1 = \{1\} \& \operatorname{Seg} 2 = \{1, 2\},\$

(5) $n = 0 \text{ or } n \in \operatorname{Seg} n,$

- $(6) n+1 \in \operatorname{Seg}(n+1),$
- (7) $n \le m \text{ iff } \operatorname{Seg} n \subseteq \operatorname{Seg} m,$
- (8) $\operatorname{Seg} n = \operatorname{Seg} m$ implies n = m,

(9)
$$k \le n \text{ implies } \operatorname{Seg} k = \operatorname{Seg} k \cap \operatorname{Seg} n \& \operatorname{Seg} k = \operatorname{Seg} n \cap \operatorname{Seg} k$$

(10)
$$\operatorname{Seg} k = \operatorname{Seg} k \cap \operatorname{Seg} n \text{ or } \operatorname{Seg} k = \operatorname{Seg} n \cap \operatorname{Seg} k \text{ implies } k \le n,$$

(11)
$$\operatorname{Seg} n \cup \{n+1\} = \operatorname{Seg} (n+1).$$

The mode

FinSequence,

which widens to the type Function, is defined by

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\mathbf{ex} n \mathbf{st} \operatorname{dom} \mathbf{it} = \operatorname{Seg} n.
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In the sequel p, q, r denote objects of the type FinSequence. Let us consider p. The functor

 $\operatorname{len} p$,

with values of the type Nat, is defined by

$$\operatorname{Seg} \mathbf{it} = \operatorname{dom} p.$$

Next we state four propositions:

(12) for f being Function holds f is FinSequence iff ex n st dom f = Seg n,

- (13) $k = \operatorname{len} p \operatorname{iff} \operatorname{Seg} k = \operatorname{dom} p,$
- (14) \emptyset is FinSequence,

(15)
$$(\mathbf{ex} k \mathbf{st} \operatorname{dom} f \subseteq \operatorname{Seg} k)$$
 implies $\mathbf{ex} p \mathbf{st} \operatorname{graph} f \subseteq \operatorname{graph} p$

In the article we present several logical schemes. The scheme SeqEx concerns a constant \mathcal{A} that has the type Nat and a binary predicate \mathcal{P} and states that the following holds

$$\mathbf{ex} p \mathbf{st} \operatorname{dom} p = \operatorname{Seg} \mathcal{A} \& \mathbf{for} k \mathbf{st} k \in \operatorname{Seg} \mathcal{A} \mathbf{holds} \mathcal{P}[k, p.k]$$

provided the parameters satisfy the following conditions:

• for k,y1,y2 st $k \in \text{Seg } \mathcal{A} \& \mathcal{P}[k,y1] \& \mathcal{P}[k,y2]$ holds y1 = y2,

for
$$k$$
 st $k \in \text{Seg } \mathcal{A} \text{ ex } x$ st $\mathcal{P}[k, x]$.

The scheme SeqLambda deals with a constant \mathcal{A} that has the type Nat and a unary functor \mathcal{F} and states that the following holds

ex p being FinSequence st len $p = \mathcal{A}$ & for k st $k \in \text{Seg }\mathcal{A}$ holds $p.k = \mathcal{F}(k)$

for all values of the parameters.

•

We now state several propositions:

(16)
$$z \in \operatorname{graph} p$$
 implies ex k st $k \in \operatorname{dom} p \& z = \langle k, p.k \rangle$,

(17) $X = \operatorname{dom} p \& X = \operatorname{dom} q \& (\operatorname{for} k \operatorname{st} k \in X \operatorname{holds} p.k = q.k) \operatorname{implies} p = q,$

(18) for
$$p,q$$

st $\operatorname{len} p = \operatorname{len} q$ & for k st $1 \le k$ & $k \le \operatorname{len} p$ holds $p \cdot k = q \cdot k$ holds p = q,

(19)
$$p \mid (\operatorname{Seg} n)$$
 is FinSequence,

(20)
$$\operatorname{rng} p \subseteq \operatorname{dom} f \text{ implies } f \cdot p \text{ is FinSequence},$$

(21)
$$k \le \operatorname{len} p \& q = p \mid (\operatorname{Seg} k) \text{ implies } \operatorname{len} q = k \& \operatorname{dom} q = \operatorname{Seg} k.$$

Let D have the type DOMAIN. The mode

FinSequence of
$$D$$
,

which widens to the type FinSequence, is defined by

rng
$$\mathbf{it} \subseteq D$$
.

In the sequel D will have the type DOMAIN. The following three propositions are true:

(22)
$$p$$
 is FinSequence of D iff rng $p \subseteq D$,

(23) for D,k for p being FinSequence of D holds $p \mid (\text{Seg } k)$ is FinSequence of D,

(24)
$$\mathbf{ex} p \mathbf{being}$$
 FinSequence of $D \mathbf{st} \operatorname{len} p = k$.

The constant ε has the type FinSequence, and is defined by

$$\operatorname{len} \mathbf{it} = 0.$$

The following propositions are true:

(25)
$$p = \varepsilon \text{ iff } \ln p = 0,$$

(26)
$$p = \varepsilon \operatorname{iff} \operatorname{dom} p = \emptyset,$$

(27)
$$p = \varepsilon \operatorname{iff} \operatorname{rng} p = \emptyset,$$

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(28)
$$\operatorname{graph} \varepsilon = \emptyset,$$

(29) for
$$D$$
 holds ε is FinSequence of D .

Let D have the type DOMAIN. The functor

 εD ,

yields the type FinSequence of D and is defined by

 $\mathbf{it} = \varepsilon$.

One can prove the following four propositions:

(30)
$$p = \varepsilon (D) \text{ iff } \operatorname{dom} p = \emptyset,$$

(31)
$$\varepsilon(D) = \varepsilon,$$

(32)
$$p = \varepsilon (D) \text{ iff } \ln p = 0,$$

(33)
$$p = \varepsilon (D) \text{ iff } \operatorname{rng} p = \emptyset.$$

Let us consider p, q. The functor

$$p \frown q$$
,

with values of the type FinSequence, is defined by

$$\operatorname{dom} \mathbf{it} = \operatorname{Seg} \left(\operatorname{len} p + \operatorname{len} q \right) \, \&$$

(for k st $k \in \text{dom } p$ holds it.k = p.k) & for k st $k \in \text{dom } q$ holds it.(len p + k) = q.k.

One can prove the following propositions:

(34)
$$r = p \cap q \text{ iff } \operatorname{dom} r = \operatorname{Seg} (\operatorname{len} p + \operatorname{len} q) \&$$
$$(\text{for } k \text{ st } k \in \operatorname{dom} p \text{ holds } r.k = p.k)$$

& for k st $k \in \operatorname{dom} q$ holds $r.(\operatorname{len} p + k) = q.k$,

(35)
$$\ln\left(p \frown q\right) = \ln p + \ln q,$$

(36) for
$$k$$
 st len $p + 1 \le k \& k \le \text{len } p + \text{len } q$ holds $(p \frown q) . k = q . (k - \text{len } p),$

(38)
$$k \in \operatorname{dom}(p \cap q)$$
 implies $k \in \operatorname{dom} p$ or $\operatorname{ex} n$ st $n \in \operatorname{dom} q$ & $k = \operatorname{len} p + n$,

(39)
$$\operatorname{dom} p \subseteq \operatorname{dom} (p \frown q),$$

(40)
$$x \in \operatorname{dom} q$$
 implies ex k st $k = x \& \operatorname{len} p + k \in \operatorname{dom} (p \cap q)$,

(41)
$$k \in \operatorname{dom} q \text{ implies } \operatorname{len} p + k \in \operatorname{dom} (p \cap q),$$

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(42)
$$\operatorname{rng} p \subseteq \operatorname{rng} (p \frown q),$$

(43)
$$\operatorname{rng} q \subseteq \operatorname{rng} (p \frown q),$$

(44)
$$\operatorname{rng}(p \frown q) = \operatorname{rng} p \cup \operatorname{rng} q,$$

$$(45) p \cap q \cap r = p \cap (q \cap r),$$

(46)
$$p \cap r = q \cap r \text{ or } r \cap p = r \cap q \text{ implies } p = q,$$

$$(47) p^{\frown} \varepsilon = p \& \varepsilon^{\frown} p = p,$$

(48)
$$p \cap q = \varepsilon$$
 implies $p = \varepsilon \& q = \varepsilon$.

The arguments of the notions defined below are the following: D which is an object of the type reserved above; p, q which are objects of the type FinSequence of D. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$p \cap q$$
 is FinSequence of D.

One can prove the following proposition

(49) for p,q being FinSequence of D holds $p \cap q$ is FinSequence of D.

Let us consider x. The functor

 $\langle x \rangle$,

with values of the type FinSequence, is defined by

dom
$$\mathbf{it} = \operatorname{Seg} 1 \& \mathbf{it} . 1 = x.$$

The following proposition is true

(50)

$p \frown q$ is FinSequence of D

implies p is FinSequence of D & q is FinSequence of D.

We now define two new functors. Let us consider x, y. The functor

$$\langle x, y \rangle,$$

with values of the type FinSequence, is defined by

$$\mathbf{it} = \langle x \rangle \cap \langle y \rangle.$$

Let us consider z. The functor

$$\langle x, y, z \rangle,$$

with values of the type FinSequence, is defined by

$$\mathbf{it} = \langle x \rangle \frown \langle y \rangle \frown \langle z \rangle.$$

Next we state a number of propositions:

(51) $p = \langle x \rangle$ iff dom p = Seg 1 & p.1 = x,

(52) $\operatorname{graph} \langle x \rangle = \{ \langle 1, x \rangle \},\$

 $(53) \qquad \qquad < x, y > = < x > ^{\frown} < y >,$

$$(54) \qquad \qquad < x, y, z > = < x > ^ < y > ^ < z >$$

(55) $p = \langle x \rangle \operatorname{iff} \operatorname{dom} p = \operatorname{Seg} 1 \& \operatorname{rng} p = \{x\},$

(56)
$$p = \langle x \rangle$$
 iff len $p = 1 \& \operatorname{rng} p = \{x\},$

(57)
$$p = \langle x \rangle$$
 iff len $p = 1 \& p.1 = x$

(58)
$$(\langle x \rangle \frown p).1 = x,$$

(59)
$$(p \frown).(\ln p + 1) = x$$

$$(60) \qquad \qquad < x, y, z > = < x > ^{\frown} < y, z > \& < x, y, z > = < x, y > ^{\frown} < z >,$$

(61)
$$p = \langle x, y \rangle$$
 iff $\operatorname{len} p = 2 \& p.1 = x \& p.2 = y$,

(62)
$$p = \langle x, y, z \rangle$$
 iff $\operatorname{len} p = 3 \& p.1 = x \& p.2 = y \& p.3 = z$,

(63) for
$$p$$
 st $p \neq \varepsilon$ ex q, x st $p = q \frown \langle x \rangle$.

The arguments of the notions defined below are the following: D which is an object of the type reserved above; x which is an object of the type Element of D. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\langle x \rangle$$
 is FinSequence of D.

The arguments of the notions defined below are the following: D which is an object of the type reserved above; S which is an object of the type SUBDOMAIN of D; x which is an object of the type Element of S. Let us note that it makes sense to consider the following functor on a restricted area. Then

$$\langle x \rangle$$
 is FinSequence of S

The arguments of the notions defined below are the following: S which is an object of the type SUBDOMAIN of REAL; x which is an object of the type Element of S. Let us note that it makes sense to consider the following functor on a restricted area. Then

 $\langle x \rangle$ is FinSequence of S.

The scheme IndSeq concerns a unary predicate \mathcal{P} states that the following holds

for p holds $\mathcal{P}[p]$

 $\mathcal{P}[\varepsilon],$

provided the parameter satisfies the following conditions:

for
$$p, x$$
 st $\mathcal{P}[p]$ holds $\mathcal{P}[p \frown \langle x \rangle]$.

One can prove the following proposition

(64)

•

for
$$p,q,r,s$$
 being FinSequence

st $p \cap q = r \cap s$ & len $p \leq \text{len } r \text{ ex } t$ being FinSequence st $p \cap t = r$.

Let us consider D. The functor

 D^* ,

yields the type DOMAIN and is defined by

 $x \in \mathbf{it} \mathbf{iff} x \mathbf{is}$ FinSequence of D.

One can prove the following propositions:

(65)
$$x \in D^*$$
 iff x is FinSequence of D

(66)
$$\varepsilon \in D^*$$
.

The scheme SepSeq deals with a constant \mathcal{A} that has the type DOMAIN and a unary predicate \mathcal{P} and states that the following holds

ex X st for x holds
$$x \in X$$
 iff ex p st $p \in \mathcal{A}^* \& \mathcal{P}[p] \& x = p$

for all values of the parameters.

The mode

FinSubsequence,

which widens to the type Function, is defined by

$$\mathbf{ex} k \mathbf{st} \operatorname{dom} \mathbf{it} \subseteq \operatorname{Seg} k.$$

The following three propositions are true:

(67) f is FinSubsequence iff ex k st dom $f \subseteq \operatorname{Seg} k$,

(68) for p being FinSequence holds p is FinSubsequence,

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(69) for p, X holds p \mid X is FinSubsequence & X \mid p is FinSubsequence.
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In the sequel p' has the type FinSubsequence. Let us consider X. Assume there exists k, such that

$$X \subseteq \operatorname{Seg} k.$$

The functor

 $\operatorname{Sgm} X,$

with values of the type FinSequence of NAT, is defined by

$$\operatorname{rng} \mathbf{it} = X \&$$

 $\mathbf{for}\, l, m, k1, k2 \; \mathbf{st}\; 1 \leq l \; \& \; l < m \; \& \; m \leq \mathrm{len}\, \mathbf{it}\; \& \; k1 = \mathbf{it}.l \; \& \; k2 = \mathbf{it}.m \; \mathbf{holds}\; k1 < k2.$

One can prove the following propositions:

(70) (ex
$$k$$
 st $X \subseteq \text{Seg } k$) implies for p being FinSequence of NAT holds
 $p = \text{Sgm } X$ iff $\text{rng } p = X \& \text{ for } l, m, k1, k2$

st
$$1 \le l \& l < m \& m \le \ln p \& k1 = p.l \& k2 = p.m$$
 holds $k1 < k2$,

(71)
$$\operatorname{rng}\operatorname{Sgm}\operatorname{dom} p' = \operatorname{dom} p'.$$

Let us consider p'. The functor

 $\operatorname{Seq} p',$

yields the type FinSequence and is defined by

$$\mathbf{it} = p' \cdot \operatorname{Sgm} (\operatorname{dom} p').$$

Next we state two propositions:

(72) **for** X **st ex** k **st** X
$$\subseteq$$
 Seg k **holds** Sgm X = ε **iff** X = \emptyset ,

(73)
$$p = \operatorname{Seq} p' \text{ iff } p = p' \cdot \operatorname{Sgm} (\operatorname{dom} p').$$

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