

Bases and Refinements of Topologies¹

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The articles [15], [7], [21], [11], [22], [5], [9], [16], [24], [23], [4], [6], [20], [1], [2], [3], [13], [8], [12], [18], [10], [19], [14], and [17] provide the notation and terminology for this paper.

1. SUBSETS AS NETS

In this article we present several logical schemes. The scheme *FraenkelInvolution* deals with a non empty set \mathcal{A} , subsets \mathcal{B} , \mathcal{C} of \mathcal{A} , and a unary functor \mathcal{F} yielding an element of \mathcal{A} , and states that:

$$\mathcal{B} = \{\mathcal{F}(a); a \text{ ranges over elements of } \mathcal{A} : a \in \mathcal{C}\}$$

provided the following conditions are satisfied:

- $\mathcal{C} = \{\mathcal{F}(a); a \text{ ranges over elements of } \mathcal{A} : a \in \mathcal{B}\}$, and
- For every element a of \mathcal{A} holds $\mathcal{F}(\mathcal{F}(a)) = a$.

The scheme *FraenkelComplement1* deals with a non empty relational structure \mathcal{A} , a family \mathcal{B} of subsets of \mathcal{A} , a set \mathcal{C} , and a unary functor \mathcal{F} yielding a subset of \mathcal{A} , and states that:

$$\mathcal{B}^c = \{\mathcal{F}(a)^c; a \text{ ranges over elements of } \mathcal{A} : a \in \mathcal{C}\}$$

provided the following condition is met:

- $\mathcal{B} = \{\mathcal{F}(a); a \text{ ranges over elements of } \mathcal{A} : a \in \mathcal{C}\}$.

The scheme *FraenkelComplement2* deals with a non empty relational structure \mathcal{A} , a family \mathcal{B} of subsets of \mathcal{A} , a set \mathcal{C} , and a unary functor \mathcal{F} yielding a subset of \mathcal{A} , and states that:

$$\mathcal{B}^c = \{\mathcal{F}(a); a \text{ ranges over elements of } \mathcal{A} : a \in \mathcal{C}\}$$

provided the following requirement is met:

- $\mathcal{B} = \{\mathcal{F}(a)^c; a \text{ ranges over elements of } \mathcal{A} : a \in \mathcal{C}\}$.

Next we state several propositions:

- (1) For every non empty relational structure R and for all elements x, y of R holds $y \in (\uparrow x)^c$ iff $x \not\leq y$.
- (2) Let S be a 1-sorted structure, T be a non empty 1-sorted structure, f be a map from S into T , and X be a subset of T . Then $(f^{-1}(X))^c = f^{-1}(X^c)$.
- (3) For every 1-sorted structure T and for every family F of subsets of T holds $F^c = \{a^c; a \text{ ranges over subsets of } T : a \in F\}$.
- (4) Let R be a non empty relational structure and F be a subset of R . Then $\uparrow F = \bigcup\{\uparrow x; x \text{ ranges over elements of } R : x \in F\}$ and $\downarrow F = \bigcup\{\downarrow x; x \text{ ranges over elements of } R : x \in F\}$.
- (5) Let R be a non empty relational structure, A be a family of subsets of R , and F be a subset of R . If $A = \{(\uparrow x)^c; x \text{ ranges over elements of } R : x \in F\}$, then $\text{Intersect}(A) = (\uparrow F)^c$.

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One can verify that there exists a FR-structure which is strict, trivial, reflexive, non empty, discrete, and finite.

One can verify that there exists a top-lattice which is strict, complete, and trivial.

Let S be a non empty relational structure and let T be an upper-bounded non empty reflexive antisymmetric relational structure. Note that there exists a map from S into T which is infs-preserving.

Let S be a non empty relational structure and let T be a lower-bounded non empty reflexive antisymmetric relational structure. One can check that there exists a map from S into T which is sups-preserving.

Let R, S be 1-sorted structures. Let us assume that the carrier of $S \subseteq$ the carrier of R . The functor $\text{incl}(S, R)$ yields a map from S into R and is defined by:

(Def. 1) $\text{incl}(S, R) = \text{id}_{\text{the carrier of } S}$.

Let R be a non empty relational structure and let S be a non empty relational substructure of R . Note that $\text{incl}(S, R)$ is monotone.

Let R be a non empty relational structure and let X be a non empty subset of R . The functor $\langle X; \text{id} \rangle$ yields a strict non empty net structure over R and is defined by:

(Def. 2) $\langle X; \text{id} \rangle = \text{incl}(\text{sub}(X), R) \cdot \langle \text{sub}(X); \text{id} \rangle$.

The functor $\langle X^{\text{op}}; \text{id} \rangle$ yields a strict non empty net structure over R and is defined as follows:

(Def. 3) $\langle X^{\text{op}}; \text{id} \rangle = \text{incl}(\text{sub}(X), R) \cdot \langle (\text{sub}(X))^{\text{op}}; \text{id} \rangle$.

We now state two propositions:

(6) Let R be a non empty relational structure and X be a non empty subset of R . Then

- (i) the carrier of $\langle X; \text{id} \rangle = X$,
- (ii) $\langle X; \text{id} \rangle$ is a full relational substructure of R , and
- (iii) for every element x of $\langle X; \text{id} \rangle$ holds $\langle X; \text{id} \rangle(x) = x$.

(7) Let R be a non empty relational structure and X be a non empty subset of R . Then

- (i) the carrier of $\langle X^{\text{op}}; \text{id} \rangle = X$,
- (ii) $\langle X^{\text{op}}; \text{id} \rangle$ is a full relational substructure of R^{op} , and
- (iii) for every element x of $\langle X^{\text{op}}; \text{id} \rangle$ holds $\langle X^{\text{op}}; \text{id} \rangle(x) = x$.

Let R be a non empty reflexive relational structure and let X be a non empty subset of R . Note that $\langle X; \text{id} \rangle$ is reflexive and $\langle X^{\text{op}}; \text{id} \rangle$ is reflexive.

Let R be a non empty transitive relational structure and let X be a non empty subset of R . Note that $\langle X; \text{id} \rangle$ is transitive and $\langle X^{\text{op}}; \text{id} \rangle$ is transitive.

Let R be a non empty reflexive relational structure and let I be a directed subset of R . Note that $\text{sub}(I)$ is directed.

Let R be a non empty reflexive relational structure and let I be a directed non empty subset of R . One can verify that $\langle I; \text{id} \rangle$ is directed.

Let R be a non empty reflexive relational structure and let F be a filtered non empty subset of R . One can check that $\langle (\text{sub}(F))^{\text{op}}; \text{id} \rangle$ is directed.

Let R be a non empty reflexive relational structure and let F be a filtered non empty subset of R . One can verify that $\langle F^{\text{op}}; \text{id} \rangle$ is directed.

2. OPERATIONS ON FAMILIES OF OPEN SETS

We now state a number of propositions:

(8) For every topological space T such that T is empty holds the topology of $T = \{\emptyset\}$.

(9) Let T be a trivial non empty topological space. Then

- (i) the topology of $T = 2^{\text{the carrier of } T}$, and
- (ii) for every point x of T holds the carrier of $T = \{x\}$ and the topology of $T = \{\emptyset, \{x\}\}$.

- (10) Let T be a trivial non empty topological space. Then $\{\text{the carrier of } T\}$ is a basis of T and \emptyset is a prebasis of T and $\{\emptyset\}$ is a prebasis of T .
- (11) For all sets X, Y and for every family A of subsets of X such that $A = \{Y\}$ holds $\text{FinMeetCl}(A) = \{Y, X\}$ and $\text{UniCl}(A) = \{Y, \emptyset\}$.
- (12) For every set X and for all families A, B of subsets of X such that $A = B \cup \{X\}$ or $B = A \setminus \{X\}$ holds $\text{Intersect}(A) = \text{Intersect}(B)$.
- (13) For every set X and for all families A, B of subsets of X such that $A = B \cup \{X\}$ or $B = A \setminus \{X\}$ holds $\text{FinMeetCl}(A) = \text{FinMeetCl}(B)$.
- (14) Let X be a set and A be a family of subsets of X . Suppose $X \in A$. Let x be a set. Then $x \in \text{FinMeetCl}(A)$ if and only if there exists a finite non empty family Y of subsets of X such that $Y \subseteq A$ and $x = \text{Intersect}(Y)$.
- (15) For every set X and for every family A of subsets of X holds $\text{UniCl}(\text{UniCl}(A)) = \text{UniCl}(A)$.
- (16) For every set X and for every empty family A of subsets of X holds $\text{UniCl}(A) = \{\emptyset\}$.
- (17) For every set X and for every empty family A of subsets of X holds $\text{FinMeetCl}(A) = \{X\}$.
- (18) For every set X and for every family A of subsets of X such that $A = \{\emptyset, X\}$ holds $\text{UniCl}(A) = A$ and $\text{FinMeetCl}(A) = A$.
- (19) Let X, Y be sets, A be a family of subsets of X , and B be a family of subsets of Y . Then
- (i) if $A \subseteq B$, then $\text{UniCl}(A) \subseteq \text{UniCl}(B)$, and
 - (ii) if $A = B$, then $\text{UniCl}(A) = \text{UniCl}(B)$.
- (20) Let X, Y be sets, A be a family of subsets of X , and B be a family of subsets of Y . If $A = B$ and $X \in A$ and $X \subseteq Y$, then $\text{FinMeetCl}(B) = \{Y\} \cup \text{FinMeetCl}(A)$.
- (21) For every set X and for every family A of subsets of X holds $\text{UniCl}(\text{FinMeetCl}(\text{UniCl}(A))) = \text{UniCl}(\text{FinMeetCl}(A))$.

3. BASES

One can prove the following propositions:

- (22) Let T be a topological space and K be a family of subsets of T . Then the topology of $T = \text{UniCl}(K)$ if and only if K is a basis of T .
- (23) Let T be a topological space and K be a family of subsets of T . Then K is a prebasis of T if and only if $\text{FinMeetCl}(K)$ is a basis of T .
- (24) Let T be a non empty topological space and B be a family of subsets of T . If $\text{UniCl}(B)$ is a prebasis of T , then B is a prebasis of T .
- (25) Let T_1, T_2 be topological spaces and B be a basis of T_1 . Suppose the carrier of T_1 is the carrier of T_2 and B is a basis of T_2 . Then the topology of T_1 is the topology of T_2 .
- (26) Let T_1, T_2 be topological spaces and P be a prebasis of T_1 . Suppose the carrier of T_1 is the carrier of T_2 and P is a prebasis of T_2 . Then the topology of T_1 is the topology of T_2 .
- (27) For every topological space T holds every basis of T is open and a prebasis of T .
- (28) For every topological space T holds every prebasis of T is open.
- (29) Let T be a non empty topological space and B be a prebasis of T . Then $B \cup \{\text{the carrier of } T\}$ is a prebasis of T .

- (30) For every topological space T and for every basis B of T holds $B \cup \{\text{the carrier of } T\}$ is a basis of T .
- (31) Let T be a topological space, B be a basis of T , and A be a subset of T . Then A is open if and only if for every point p of T such that $p \in A$ there exists a subset a of T such that $a \in B$ and $p \in a$ and $a \subseteq A$.
- (32) Let T be a topological space and B be a family of subsets of T . Suppose that
- $B \subseteq$ the topology of T , and
 - for every subset A of T such that A is open and for every point p of T such that $p \in A$ there exists a subset a of T such that $a \in B$ and $p \in a$ and $a \subseteq A$.
- Then B is a basis of T .
- (33) Let S be a topological space, T be a non empty topological space, K be a basis of T , and f be a map from S into T . Then f is continuous if and only if for every subset A of T such that $A \in K$ holds $f^{-1}(A^c)$ is closed.
- (34) Let S be a topological space, T be a non empty topological space, K be a basis of T , and f be a map from S into T . Then f is continuous if and only if for every subset A of T such that $A \in K$ holds $f^{-1}(A)$ is open.
- (35) Let S be a topological space, T be a non empty topological space, K be a prebasis of T , and f be a map from S into T . Then f is continuous if and only if for every subset A of T such that $A \in K$ holds $f^{-1}(A^c)$ is closed.
- (36) Let S be a topological space, T be a non empty topological space, K be a prebasis of T , and f be a map from S into T . Then f is continuous if and only if for every subset A of T such that $A \in K$ holds $f^{-1}(A)$ is open.
- (37) Let T be a non empty topological space, x be a point of T , X be a subset of T , and K be a basis of T . Suppose that for every subset A of T such that $A \in K$ and $x \in A$ holds A meets X . Then $x \in \bar{X}$.
- (38) Let T be a non empty topological space, x be a point of T , X be a subset of T , and K be a prebasis of T . Suppose that for every finite family Z of subsets of T such that $Z \subseteq K$ and $x \in \text{Intersect}(Z)$ holds $\text{Intersect}(Z)$ meets X . Then $x \in \bar{X}$.
- (39) Let T be a non empty topological space, K be a prebasis of T , x be a point of T , and N be a net in T . Suppose that for every subset A of T such that $A \in K$ and $x \in A$ holds N is eventually in A . Let S be a subset of T . If $\text{rng netmap}(N, T) \subseteq S$, then $x \in \bar{S}$.

4. PRODUCT TOPOLOGIES

We now state four propositions:

- (40) Let T_1, T_2 be non empty topological spaces, B_1 be a basis of T_1 , and B_2 be a basis of T_2 . Then $\{[a, b]; a \text{ ranges over subsets of } T_1, b \text{ ranges over subsets of } T_2: a \in B_1 \wedge b \in B_2\}$ is a basis of $[T_1, T_2]$.
- (41) Let T_1, T_2 be non empty topological spaces, B_1 be a prebasis of T_1 , and B_2 be a prebasis of T_2 . Then $\{[\text{the carrier of } T_1, b]; b \text{ ranges over subsets of } T_2: b \in B_2\} \cup \{[a, \text{the carrier of } T_2]; a \text{ ranges over subsets of } T_1: a \in B_1\}$ is a prebasis of $[T_1, T_2]$.
- (42) Let X_1, X_2 be sets, A be a family of subsets of $[X_1, X_2]$, A_1 be a non empty family of subsets of X_1 , and A_2 be a non empty family of subsets of X_2 . Suppose $A = \{[a, b]; a \text{ ranges over subsets of } X_1, b \text{ ranges over subsets of } X_2: a \in A_1 \wedge b \in A_2\}$. Then $\text{Intersect}(A) = [\text{Intersect}(A_1), \text{Intersect}(A_2)]$.
- (43) Let T_1, T_2 be non empty topological spaces, B_1 be a prebasis of T_1 , and B_2 be a prebasis of T_2 . Suppose $\bigcup B_1 = \text{the carrier of } T_1$ and $\bigcup B_2 = \text{the carrier of } T_2$. Then $\{[a, b]; a \text{ ranges over subsets of } T_1, b \text{ ranges over subsets of } T_2: a \in B_1 \wedge b \in B_2\}$ is a prebasis of $[T_1, T_2]$.

5. TOPOLOGICAL AUGMENTATIONS

Let R be a relational structure. A FR-structure is called a topological augmentation of R if:

(Def. 4) The relational structure of it = the relational structure of R .

Let R be a relational structure and let T be a topological augmentation of R . We introduce T is correct as a synonym of T is topological space-like.

Let R be a relational structure. Note that there exists a topological augmentation of R which is correct, discrete, and strict.

One can prove the following propositions:

- (44) Every FR-structure T is a topological augmentation of T .
- (45) Let S be a FR-structure and T be a topological augmentation of S . Then S is a topological augmentation of T .
- (46) Let R be a relational structure and T_1 be a topological augmentation of R . Then every topological augmentation of T_1 is a topological augmentation of R .

Let R be a non empty relational structure. Note that every topological augmentation of R is non empty.

Let R be a reflexive relational structure. One can check that every topological augmentation of R is reflexive.

Let R be a transitive relational structure. Observe that every topological augmentation of R is transitive.

Let R be an antisymmetric relational structure. Observe that every topological augmentation of R is antisymmetric.

Let R be a complete non empty relational structure. Observe that every topological augmentation of R is complete.

The following three propositions are true:

- (47) Let S be an up-complete antisymmetric non empty reflexive relational structure and T be a non empty reflexive relational structure. Suppose the relational structure of $S =$ the relational structure of T . Let A be a subset of S and C be a subset of T . If $A = C$ and A is inaccessible, then C is inaccessible.
- (48) Let S be a non empty reflexive relational structure and T be a topological augmentation of S . If the topology of $T = \sigma(S)$, then T is correct.
- (49) Let S be a complete lattice and T be a topological augmentation of S . If the topology of $T = \sigma(S)$, then T is Scott.

Let R be a complete lattice. Observe that there exists a topological augmentation of R which is Scott, strict, and correct.

Next we state three propositions:

- (50) Let S, T be complete Scott non empty reflexive transitive antisymmetric FR-structures. Suppose the relational structure of $S =$ the relational structure of T . Let F be a subset of S and G be a subset of T . If $F = G$, then if F is open, then G is open.
- (51) For every complete lattice S and for every Scott topological augmentation T of S holds the topology of $T = \sigma(S)$.
- (52) Let S, T be complete lattices. Suppose the relational structure of $S =$ the relational structure of T . Then $\sigma(S) = \sigma(T)$.

Let R be a complete lattice. One can verify that every topological augmentation of R which is Scott is also correct.

6. REFINEMENTS

Let T be a topological structure. A topological space is said to be a topological extension of T if:

(Def. 5) The carrier of $T =$ the carrier of it and the topology of $T \subseteq$ the topology of it.

Next we state the proposition

(53) Let S be a topological structure. Then there exists a topological extension T of S such that T is strict and the topology of S is a prebasis of T .

Let T be a topological structure. Observe that there exists a topological extension of T which is strict and discrete.

Let T_1, T_2 be topological structures. A topological space is said to be a refinement of T_1 and T_2 if it satisfies the conditions (Def. 6).

(Def. 6)(i) The carrier of it = (the carrier of T_1) \cup (the carrier of T_2), and

(ii) (the topology of T_1) \cup (the topology of T_2) is a prebasis of it.

Let T_1 be a non empty topological structure and let T_2 be a topological structure. Note that every refinement of T_1 and T_2 is non empty and every refinement of T_2 and T_1 is non empty.

We now state several propositions:

(54) Let T_1, T_2 be topological structures and T, T' be refinements of T_1 and T_2 . Then the topological structure of $T =$ the topological structure of T' .

(55) For all topological structures T_1, T_2 holds every refinement of T_1 and T_2 is a refinement of T_2 and T_1 .

(56) Let T_1, T_2 be topological structures, T be a refinement of T_1 and T_2 , and X be a family of subsets of T . Suppose $X =$ (the topology of T_1) \cup (the topology of T_2). Then the topology of $T = \text{UniCl}(\text{FinMeetCl}(X))$.

(57) Let T_1, T_2 be topological structures. Suppose the carrier of $T_1 =$ the carrier of T_2 . Then every refinement of T_1 and T_2 is a topological extension of T_1 and a topological extension of T_2 .

(58) Let T_1, T_2 be non empty topological spaces, T be a refinement of T_1 and T_2 , B_1 be a prebasis of T_1 , and B_2 be a prebasis of T_2 . Then $B_1 \cup B_2 \cup \{\text{the carrier of } T_1, \text{ the carrier of } T_2\}$ is a prebasis of T .

(59) Let T_1, T_2 be non empty topological spaces, B_1 be a basis of T_1 , B_2 be a basis of T_2 , and T be a refinement of T_1 and T_2 . Then $B_1 \cup B_2 \cup B_1 \cap B_2$ is a basis of T .

(60) Let T_1, T_2 be non empty topological spaces. Suppose the carrier of $T_1 =$ the carrier of T_2 . Let T be a refinement of T_1 and T_2 . Then (the topology of T_1) \cap (the topology of T_2) is a basis of T .

(61) Let L be a non empty relational structure, T_1, T_2 be correct topological augmentations of L , and T be a refinement of T_1 and T_2 . Then (the topology of T_1) \cap (the topology of T_2) is a basis of T .

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