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Miscellaneous Facts about Relation Structure[1]

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Summary. In the article notation and facts necessary to start with formalization of continuous lattices according to [5] are introduced.

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The articles [1], [3], [4], [2], and [6] provide the notation and terminology for this paper.

1. Introduction

The following propositions are true:

(1) For every reflexive antisymmetric relational structure L with l.u.b.'s and for every element a of L holds $a \sqcup a = a$.

(2) For every reflexive antisymmetric relational structure L with g.l.b.'s and for every element a of L holds $a \sqcap a = a$.

(3) Let L be a transitive antisymmetric relational structure with l.u.b.'s and a, b, c be elements of L. If $a \sqcup b \leq c$, then $a \leq c$.

(4) Let L be a transitive antisymmetric relational structure with g.l.b.'s and a, b, c be elements of L. If $c \leq a \sqcap b$, then $c \leq a$.

(5) Let L be an antisymmetric transitive relational structure with l.u.b.'s and g.l.b.'s and a, b, c be elements of L. Then $a \sqcap b \leq a \sqcup c$.

(6) Let L be an antisymmetric transitive relational structure with g.l.b.'s and a, b, c be elements of L. If $a \leq b$, then $a \sqcap c \leq b \sqcap c$.

(7) Let L be an antisymmetric transitive relational structure with l.u.b.'s and a, b, c be elements of L. If $a \leq b$, then $a \sqcup c \leq b \sqcup c$.

(8) For every sup-semilattice L and for all elements a, b of L such that $a \leq b$ holds $a \sqcup b = b$.

(9) For every sup-semilattice L and for all elements a, b, c of L such that $a \leq c$ and $b \leq c$ holds $a \sqcup b \leq c$.

(10) For every semilattice L and for all elements a, b of L such that $b \leq a$ holds $a \sqcap b = b$.

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2. Difference in Relation Structure

The following proposition is true

(11) For every Boolean lattice L and for all elements x, y of L holds y is a complement of x iff $y = \neg x$.

Let L be a non empty relational structure and let a, b be elements of L. The functor $a \setminus b$ yielding an element of L is defined by:

(Def. 1) $a \setminus b = a \sqcap \neg b$.

Let L be a non empty relational structure and let a, b be elements of L. The functor $a \dot{-} b$ yields an element of L and is defined by:

(Def. 2) $a \dot{-} b = (a \setminus b) \sqcup (b \setminus a)$.

Let L be an antisymmetric relational structure with g.l.b.'s and l.u.b.'s and let a, b be elements of L. Let us note that the functor $a \dot{-} b$ is commutative.

Let L be a non empty relational structure and let a, b be elements of L. We say that a meets b if and only if:

(Def. 3) $a \sqcap b \neq \bot_L$.

We introduce a misses b as an antonym of a meets b.

Let L be an antisymmetric relational structure with g.l.b.'s and let a, b be elements of L. Let us note that the predicate a meets b is symmetric. We introduce a misses b as an antonym of a meets b.

We now state a number of propositions:

(12) Let L be an antisymmetric transitive relational structure with g.l.b.'s and l.u.b.'s and a, b, c be elements of L. If $a \leq c$, then $a \setminus b \leq c$.

(13) Let L be an antisymmetric transitive relational structure with g.l.b.'s and l.u.b.'s and a, b, c be elements of L. If $a \leq b$, then $a \setminus c \leq b \setminus c$.

(14) Let L be an antisymmetric transitive relational structure with g.l.b.'s and l.u.b.'s and a, b be elements of L. Then $a \setminus b \leq a$.

(15) Let L be an antisymmetric transitive relational structure with g.l.b.'s and l.u.b.'s and a, b be elements of L. Then $a \setminus b \leq a \dot{-} b$.

(16) For every lattice L and for all elements a, b, c of L such that $a \setminus b \leq c$ and $b \setminus a \leq c$ holds $a \dot{-} b \leq c$.

(17) For every lattice L and for every element a of L holds a meets a iff $a \neq \bot_L$.

(18) Let L be an antisymmetric transitive relational structure with g.l.b.'s and l.u.b.'s and a, b, c be elements of L. Then $a \sqcap (b \setminus c) = (a \sqcap b) \setminus c$.

(19) Let L be an antisymmetric transitive relational structure with g.l.b.'s. Suppose L is distributive. Let a, b, c be elements of L. If $(a \sqcap b) \sqcup (a \sqcap c) = a$, then $a \leq b \sqcup c$.

(20) For every lattice L such that L is distributive and for all elements a, b, c of L holds $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$.

(21) For every lattice L such that L is distributive and for all elements a, b, c of L holds $(a \sqcup b) \setminus c = (a \setminus c) \sqcup (b \setminus c)$.

3. Lower-bound in Relation Structure

We now state a number of propositions:

(22) Let L be a lower-bounded non empty antisymmetric relational structure and a be an element of L. If $a \leq \bot_L$, then $a = \bot_L$.

(23) Let L be a lower-bounded semilattice and a, b, c be elements of L. If $a \leq b$ and $a \leq c$ and $b \sqcap c = \bot_L$, then $a = \bot_L$.

(24) Let L be a lower-bounded antisymmetric relational structure with l.u.b.'s and a, b be elements of L. If $a \sqcup b = \bot_L$, then $a = \bot_L$ and $b = \bot_L$.

(25) Let L be a lower-bounded antisymmetric transitive relational structure with g.l.b.'s and a, b, c be elements of L. If $a \leq b$ and $b \sqcap c = \bot_L$, then $a \sqcap c = \bot_L$.

(26) For every lower-bounded semilattice L and for every element a of L holds $\bot_L \setminus a = \bot_L$.

(27) Let L be a lower-bounded antisymmetric transitive relational structure with g.l.b.'s and a, b, c be elements of L. If a meets b and $b \leq c$, then a meets c.

(28) Let L be a lower-bounded antisymmetric relational structure with g.l.b.'s and a be an element of L. Then $a \sqcap \bot_L = \bot_L$.

(29) Let L be a lower-bounded antisymmetric transitive relational structure with g.l.b.'s and l.u.b.'s and a, b, c be elements of L. If a meets $b \sqcap c$, then a meets b.

(30) Let L be a lower-bounded antisymmetric transitive relational structure with g.l.b.'s and l.u.b.'s and a, b, c be elements of L. If a meets $b \setminus c$, then a meets b.

(31) Let L be a lower-bounded antisymmetric transitive relational structure with g.l.b.'s and a be an element of L. Then a misses $\bot_L$.

(32) Let L be a lower-bounded antisymmetric transitive relational structure with g.l.b.'s and a, b, c be elements of L. If a misses c and $b \leq c$, then a misses b.

(33) Let L be a lower-bounded antisymmetric transitive relational structure with g.l.b.'s and a, b, c be elements of L. If a misses b or a misses c, then a misses $b \sqcap c$.

(34) Let L be a lower-bounded lattice and a, b, c be elements of L. If $a \leq b$ and $a \leq c$ and b misses c, then $a = \bot_L$.

(35) Let L be a lower-bounded antisymmetric transitive relational structure with g.l.b.'s and a, b, c be elements of L. If a misses b, then $a \sqcap c$ misses $b \sqcap c$.

4. Boolean Lattices

We adopt the following rules: L is a Boolean non empty relational structure and a, b, c, d are elements of L.

Next we state a number of propositions:

(36) $(a \sqcap b) \sqcup (b \sqcap c) \sqcup (c \sqcap a) = (a \sqcup b) \sqcap (b \sqcup c) \sqcap (c \sqcup a)$.

(37) $a \sqcap \neg a = \bot_L$ and $a \sqcup \neg a = \top_L$.

(38) If $a \setminus b \leq c$, then $a \leq b \sqcup c$.

(39) $\neg(a \sqcup b) = \neg a \sqcap \neg b$ and $\neg(a \sqcap b) = \neg a \sqcup \neg b$.

(40) If $a \leq b$, then $\neg b \leq \neg a$.

(41) If $a \leq b$, then $c \setminus b \leq c \setminus a$.

(42) If $a \leq b$ and $c \leq d$, then $a \setminus d \leq b \setminus c$.

(43) If $a \leq b \sqcup c$, then $a \setminus b \leq c$ and $a \setminus c \leq b$.

(44) $\neg a \leq \neg(a \sqcap b)$ and $\neg b \leq \neg(a \sqcap b)$.

(45) $\neg(a \sqcup b) \leq \neg a$ and $\neg(a \sqcup b) \leq \neg b$.

(46) If $a \leq b \setminus a$, then $a = \bot_L$.

(47) If $a \leq b$, then $b = a \sqcup (b \setminus a)$.

(48) $a \setminus b = \bot_L$ iff $a \leq b$.

(49) If $a \leq b \sqcup c$ and $a \sqcap c = \bot_L$, then $a \leq b$.

(50) $a \sqcup b = (a \setminus b) \sqcup b$.

(51) $a \setminus (a \sqcup b) = \bot_L$.

(52) $a \setminus (a \sqcap b) = a \setminus b$.

(53) $(a \setminus b) \sqcap b = \bot_L$.

(54) $a \sqcup (b \setminus a) = a \sqcup b$.

(55) $(a \sqcap b) \sqcup (a \setminus b) = a$.

(56) $a \setminus (b \setminus c) = (a \setminus b) \sqcup (a \sqcap c)$.

(57) $a \setminus (a \setminus b) = a \sqcap b$.

(58) $(a \sqcup b) \setminus b = a \setminus b$.

(59) $a \sqcap b = \bot_L$ iff $a \setminus b = a$.

(60) $a \setminus (b \sqcup c) = (a \setminus b) \sqcap (a \setminus c)$.

(61) $a \setminus (b \sqcap c) = (a \setminus b) \sqcup (a \setminus c)$.

(62) $a \sqcap (b \setminus c) = (a \sqcap b) \setminus (a \sqcap c)$.

(63) $(a \sqcup b) \setminus (a \sqcap b) = (a \setminus b) \sqcup (b \setminus a)$.

(64) $a \setminus b \setminus c = a \setminus (b \sqcup c)$.

(65) $\neg(\bot_L) = \top_L$.

(66) $\neg(\top_L) = \bot_L$.

(67) $a \setminus a = \bot_L$.

(68) $a \setminus \bot_L = a$.

(69) $\neg(a \setminus b) = \neg a \sqcup b$.

(70) $a \sqcap b$ misses $a \setminus b$.

(71) $a \setminus b$ misses b.

(72) If a misses b, then $(a \sqcup b) \setminus b = a$.

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