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Continuous Lattices of Maps between T_0 Spaces[1]

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The articles [32], [15], [38], [33], [19], [39], [13], [40], [18], [12], [16], [1], [10], [30], [2], [35], [27], [28], [29], [37], [3], [14], [31], [24], [11], [42], [34], [4], [5], [6], [22], [41], [17], [8], [7], [25], [36], [23], [21], [26], and [9] provide the notation and terminology for this paper.

Let I be a set and let J be a relational structure yielding many sorted set indexed by I. We introduce $I\text{-prod}_{\text{POS}} J$ as a synonym of $\prod J$.

Let I be a set and let J be a relational structure yielding nonempty many sorted set indexed by I. Observe that $I\text{-prod}_{\text{POS}} J$ is constituted functions.

Let I be a set and let J be a topological space yielding nonempty many sorted set indexed by I. We introduce $I\text{-prod}_{\text{TOP}} J$ as a synonym of $\prod J$.

Let X, Y be non empty topological spaces. The functor $[X \to Y]$ yields a non empty strict relational structure and is defined as follows:

(Def. 1) $[X \to Y] = [X \to \Omega Y]$.

Let X, Y be non empty topological spaces. One can check that $[X \to Y]$ is reflexive, transitive, and constituted functions.

Let X be a non empty topological space and let Y be a non empty T_0 topological space. Observe that $[X \to Y]$ is antisymmetric.

One can prove the following three propositions:

(1) Let X, Y be non empty topological spaces and a be a set. Then a is an element of $[X \to Y]$ if and only if a is a continuous map from X into ΩY.

(2) Let X, Y be non empty topological spaces and a be a set. Then a is an element of $[X \to Y]$ if and only if a is a continuous map from X into Y.

(3) Let X, Y be non empty topological spaces, a, b be elements of $[X \to Y]$, and f, g be maps from X into ΩY. If $a = f$ and $b = g$, then $a \leq b$ iff $f \leq g$.

Let X, Y be non empty topological spaces, let x be a point of X, and let A be a subset of $[X \to Y]$. Then $\pi_x A$ is a subset of ΩY.

Let X, Y be non empty topological spaces, let x be a set, and let A be a non empty subset of $[X \to Y]$. Observe that $\pi_x A$ is non empty.

The following propositions are true:

(4) Ω(the Sierpiński space) is a topological augmentation of $2^1_\subseteq$.

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(5) Let X be a non empty topological space. Then there exists a map f from $\langle$the topology of X, $\subseteq\rangle$ into $[X \rightarrow$ the Sierpiński space$]$ such that f is isomorphic and for every open subset V of X holds $f(V) = \chi_{V,\text{the carrier of } X}$.

(6) Let X be a non empty topological space. Then $\langle$the topology of X, $\subseteq\rangle$ and $[X \rightarrow$ the Sierpiński space$]$ are isomorphic.

Let X, Y, Z be non empty topological spaces and let f be a continuous map from Y into Z. The functor $[X \rightarrow f]$ yields a map from $[X \rightarrow Y]$ into $[X \rightarrow Z]$ and is defined by:

(Def. 2) For every continuous map g from X into Y holds $([X \rightarrow f])(g) = f \cdot g$.

The functor $[f \rightarrow X]$ yielding a map from $[Z \rightarrow X]$ into $[Y \rightarrow X]$ is defined by:

(Def. 3) For every continuous map g from Z into X holds $([f \rightarrow X])(g) = g \cdot f$.

We now state a number of propositions:

(7) Let X be a non empty topological space and Y be a monotone convergence T_0-space. Then $[X \rightarrow Y]$ is a directed-sups-inheriting relational substructure of $(\Omega Y)^{\text{the carrier of } X}$.

(8) For every non empty topological space X and for every monotone convergence T_0-space Y holds $[X \rightarrow Y]$ is up-complete.

(9) For all non empty topological spaces X, Y, Z and for every continuous map f from Y into Z holds $[X \rightarrow f]$ is monotone.

(10) Let X, Y be non empty topological spaces and f be a continuous map from Y into Y. If f is idempotent, then $[X \rightarrow f]$ is idempotent.

(11) For all non empty topological spaces X, Y, Z and for every continuous map f from Y into Z holds $[f \rightarrow X]$ is monotone.

(12) Let X, Y be non empty topological spaces and f be a continuous map from Y into Y. If f is idempotent, then $[f \rightarrow X]$ is idempotent.

(13) Let X, Y, Z be non empty topological spaces, f be a continuous map from Y into Z, x be an element of X, and A be a subset of $[X \rightarrow Y]$. Then $\pi_x([X \rightarrow f])^\circ A = f^\circ \pi_x A$.

(14) Let X be a non empty topological space, Y, Z be monotone convergence T_0-spaces, and f be a continuous map from Y into Z. Then $[X \rightarrow f]$ is directed-sups-preserving.

(15) Let X, Y, Z be non empty topological spaces, f be a continuous map from Y into Z, x be an element of Y, and A be a subset of $[Z \rightarrow X]$. Then $\pi_x([f \rightarrow X])^\circ A = \pi_{f(x)} A$.

(16) Let Y, Z be non empty topological spaces, X be a monotone convergence T_0-space, and f be a continuous map from Y into Z. Then $[f \rightarrow X]$ is directed-sups-preserving.

(17) Let X, Z be non empty topological spaces and Y be a non empty subspace of Z. Then $[X \rightarrow Y]$ is a full relational substructure of $[X \rightarrow Z]$.

(18) Let Z be a monotone convergence T_0-space, Y be a non empty subspace of Z, and f be a continuous map from Z into Y. Suppose f is a retraction. Then ΩY is a directed-sups-inheriting relational substructure of ΩZ.

(19) Let X be a non empty topological space, Z be a monotone convergence T_0-space, Y be a non empty subspace of Z, and f be a continuous map from Z into Y. If f is a retraction, then $[X \rightarrow f]$ is a retraction of $[X \rightarrow Z]$ into $[X \rightarrow Y]$.

(20) Let X be a non empty topological space, Z be a monotone convergence T_0-space, and Y be a non empty subspace of Z. If Y is a retract of Z, then $[X \rightarrow Y]$ is a retract of $[X \rightarrow Z]$.

(21) Let X, Y, Z be non empty topological spaces and f be a continuous map from Y into Z. If f is a homeomorphism, then $[X \to f]$ is isomorphic.

(22) Let X, Y, Z be non empty topological spaces. If Y and Z are homeomorphic, then $[X \to Y]$ and $[X \to Z]$ are isomorphic.

(23) Let X be a non empty topological space, Z be a monotone convergence T_0-space, and Y be a non empty subspace of Z. Suppose Y is a retract of Z and $[X \to Z]$ is complete and continuous. Then $[X \to Y]$ is complete and continuous.

(24) Let X be a non empty topological space and Y, Z be monotone convergence T_0-spaces. Suppose Y is a topological retract of Z and $[X \to Z]$ is complete and continuous. Then $[X \to Y]$ is complete and continuous.

(25) Let Y be a non trivial T_0-space. Suppose Y is not a T_1 space. Then the Sierpiński space is a topological retract of Y.

(26) Let X be a non empty topological space and Y be a non trivial T_0-space. If $[X \to Y]$ has l.u.b.'s, then Y is not a T_1 space.

Let us observe that the Sierpiński space is non trivial and monotone convergence.

Let us observe that there exists a T_0-space which is injective, monotone convergence, and non trivial.

One can prove the following propositions:

(27) Let X be a non empty topological space and Y be a monotone convergence non trivial T_0-space. If $[X \to Y]$ is complete and continuous, then $\langle$the topology of X, $\subseteq\rangle$ is continuous.

(28) Let X be a non empty topological space, x be a point of X, and Y be a monotone convergence T_0-space. Then there exists a directed-sups-preserving projection map F from $[X \to Y]$ into $[X \to Y]$ such that

(i) for every continuous map f from X into Y holds $F(f) = X \longmapsto f(x)$, and

(ii) there exists a continuous map h from X into X such that $h = X \longmapsto x$ and $F = [h \to Y]$.

(29) Let X be a non empty topological space and Y be a monotone convergence T_0-space. If $[X \to Y]$ is complete and continuous, then ΩY is complete and continuous.

(30) Let X be a non empty 1-sorted structure, I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, f be a map from X into I-$\mathrm{prod}_{\mathrm{TOP}} J$, and i be an element of I. Then $(\mathrm{commute}(f))(i) = \mathrm{proj}(J, i) \cdot f$.

(31) For every 1-sorted structure S and for every set M holds the support of $M \longmapsto S = M \longmapsto$ the carrier of S.

(32) Let X, Y be non empty topological spaces, M be a non empty set, and f be a continuous map from X into M-$\mathrm{prod}_{\mathrm{TOP}}(M \longmapsto Y)$. Then $\mathrm{commute}(f)$ is a function from M into the carrier of $([X \to Y])$.

(33) For all non empty topological spaces X, Y holds the carrier of $([X \to Y]) \subseteq$ (the carrier of $Y)^{\text{the carrier of } X}$.

(34) Let X, Y be non empty topological spaces, M be a non empty set, and f be a function from M into the carrier of $([X \to Y])$. Then $\mathrm{commute}(f)$ is a continuous map from X into M-$\mathrm{prod}_{\mathrm{TOP}}(M \longmapsto Y)$.

(35) Let X be a non empty topological space and M be a non empty set. Then there exists a map F from $[X \to M$-$\mathrm{prod}_{\mathrm{TOP}}(M \longmapsto$ the Sierpiński space$)]$ into M-$\mathrm{prod}_{\mathrm{POS}}(M \longmapsto ([X \to$ the Sierpiński space$]))$ such that F is isomorphic and for every continuous map f from X into M-$\mathrm{prod}_{\mathrm{TOP}}(M \longmapsto$ the Sierpiński space$)$ holds $F(f) = \mathrm{commute}(f)$.

(36) Let X be a non empty topological space and M be a non empty set. Then $[X \to M\text{-prod}_{\rm TOP}(M \longmapsto \text{the Sierpiński space})]$ and $M\text{-prod}_{\rm POS}(M \longmapsto ([X \to \text{the Sierpiński space}]))$ are isomorphic.

(37) Let X be a non empty topological space. Suppose $\langle$the topology of X, $\subseteq\rangle$ is continuous. Let Y be an injective T_0-space. Then $[X \to Y]$ is complete and continuous.

Let us mention that there exists a top-lattice which is non trivial, complete, and Scott.
The following proposition is true

(38) Let X be a non empty topological space and L be a non trivial complete Scott top-lattice. Then $[X \to L]$ is complete and continuous if and only if $\langle$the topology of X, $\subseteq\rangle$ is continuous and L is continuous.

Let f be a function. One can verify that $\bigcup$disjoint f is relation-like.
Let f be a function. The functor G_f yields a binary relation and is defined as follows:

(Def. 4) $G_f = (\bigcup \text{disjoint} f)^{\smile}$.

In the sequel x, y are sets and f is a function.
One can prove the following three propositions:

(39) $\langle x, y\rangle \in G_f$ iff $x \in \operatorname{dom} f$ and $y \in f(x)$.

(40) For every finite set X holds $\pi_1(X)$ is finite and $\pi_2(X)$ is finite.

(41) Let X, Y be non empty topological spaces, S be a Scott topological augmentation of $\langle$the topology of Y, $\subseteq\rangle$, and f be a map from X into S. If G_f is an open subset of $[:X, Y:]$, then f is continuous.

Let W be a binary relation and let X be a set. The functor $\Theta_X(W)$ yielding a function is defined as follows:

(Def. 5) $\operatorname{dom}\Theta_X(W) = X$ and for every x such that $x \in X$ holds $(\Theta_X(W))(x) = W^\circ\{x\}$.

We now state the proposition

(42) For every binary relation W and for every set X such that $\operatorname{dom} W \subseteq X$ holds $G_{\Theta_X(W)} = W$.

Let X, Y be topological spaces. Observe that every subset of $[:X, Y:]$ is relation-like and every element of the topology of $[:X, Y:]$ is relation-like.
Next we state four propositions:

(43) Let X, Y be non empty topological spaces, W be an open subset of $[:X, Y:]$, and x be a point of X. Then $W^\circ\{x\}$ is an open subset of Y.

(44) Let X, Y be non empty topological spaces, S be a Scott topological augmentation of $\langle$the topology of Y, $\subseteq\rangle$, and W be an open subset of $[:X, Y:]$. Then $\Theta_{\text{the carrier of } X}(W)$ is a continuous map from X into S.

(45) Let X, Y be non empty topological spaces, S be a Scott topological augmentation of $\langle$the topology of Y, $\subseteq\rangle$, and W_1, W_2 be open subsets of $[:X, Y:]$. Suppose $W_1 \subseteq W_2$. Let f_1, f_2 be elements of $[X \to S]$. If $f_1 = \Theta_{\text{the carrier of } X}(W_1)$ and $f_2 = \Theta_{\text{the carrier of } X}(W_2)$, then $f_1 \leq f_2$.

(46) Let X, Y be non empty topological spaces and S be a Scott topological augmentation of $\langle$the topology of Y, $\subseteq\rangle$. Then there exists a map F from $\langle$the topology of $[:X, Y:]$, $\subseteq\rangle$ into $[X \to S]$ such that F is monotone and for every open subset W of $[:X, Y:]$ holds $F(W) = \Theta_{\text{the carrier of } X}(W)$.

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