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Sets and Functions of Trees and Joining Operations of Trees

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Summary. In the article we deal with sets of trees and functions yielding trees. So, we introduce the sets of all trees, all finite trees and of all trees decorated by elements from some set. Next, the functions and the finite sequences yielding (finite, decorated) trees are introduced. There are shown some convenient but technical lemmas and clusters concerning with those concepts. In the fourth section we deal with trees decorated by Cartesian product and we introduce the concept of a tree called a substitution of structure of some finite tree. Finally, we introduce the operations of joining trees, i.e. for the finite sequence of trees we define the tree which is made by joining the trees from the sequence by common root. For one and two trees there are introduced the same operations.

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The articles [14], [9], [17], [15], [1], [18], [7], [5], [11], [13], [16], [12], [19], [8], [6], [10], [2], [4], and [3] provide the notation and terminology for this paper.

1. Finite sets

For simplicity, we adopt the following rules: x, y denote sets, i, n denote natural numbers, p, q denote finite sequences, X, Y denote sets, and f denotes a function.

2. Sets of trees

The set Trees is defined as follows:

(Def. 1) $x \in$ Trees iff x is a tree.

One can check that Trees is non empty.
The subset FinTrees of Trees is defined as follows:

(Def. 2) $x \in$ FinTrees iff x is a finite tree.

Let us mention that FinTrees is non empty.
Let I_1 be a set. We say that I_1 is constituted of trees if and only if:

(Def. 3) For every x such that $x \in I_1$ holds x is a tree.

We say that I_1 is constituted of finite trees if and only if:

(Def. 4) For every x such that $x \in I_1$ holds x is a finite tree.

We say that I_1 is constituted of decorated trees if and only if:

(Def. 5) For every x such that $x \in I_1$ holds x is a decorated tree.

We now state a number of propositions:

(1) X is constituted of trees iff $X \subseteq$ Trees.

(2) X is constituted of finite trees iff $X \subseteq$ FinTrees.

(3) X is constituted of trees and Y is constituted of trees iff $X \cup Y$ is constituted of trees.

(4) If X is constituted of trees and Y is constituted of trees, then $X \dot{-} Y$ is constituted of trees.

(5) Suppose X is constituted of trees. Then $X \cap Y$ is constituted of trees and $Y \cap X$ is constituted of trees and $X \setminus Y$ is constituted of trees.

(6) X is constituted of finite trees and Y is constituted of finite trees if and only if $X \cup Y$ is constituted of finite trees.

(7) Suppose X is constituted of finite trees and Y is constituted of finite trees. Then $X \dot{-} Y$ is constituted of finite trees.

(8) Suppose X is constituted of finite trees. Then

(i) $X \cap Y$ is constituted of finite trees,

(ii) $Y \cap X$ is constituted of finite trees, and

(iii) $X \setminus Y$ is constituted of finite trees.

(9) X is constituted of decorated trees and Y is constituted of decorated trees if and only if $X \cup Y$ is constituted of decorated trees.

(10) Suppose X is constituted of decorated trees and Y is constituted of decorated trees. Then $X \dot{-} Y$ is constituted of decorated trees.

(11) Suppose X is constituted of decorated trees. Then

(i) $X \cap Y$ is constituted of decorated trees,

(ii) $Y \cap X$ is constituted of decorated trees, and

(iii) $X \setminus Y$ is constituted of decorated trees.

(12) $\emptyset$ is constituted of trees, constituted of finite trees, and constituted of decorated trees.

(13) $\{x\}$ is constituted of trees iff x is a tree.

(14) $\{x\}$ is constituted of finite trees iff x is a finite tree.

(15) $\{x\}$ is constituted of decorated trees iff x is a decorated tree.

(16) $\{x, y\}$ is constituted of trees iff x is a tree and y is a tree.

(17) $\{x, y\}$ is constituted of finite trees iff x is a finite tree and y is a finite tree.

(18) $\{x, y\}$ is constituted of decorated trees iff x is a decorated tree and y is a decorated tree.

(19) If X is constituted of trees and $Y \subseteq X$, then Y is constituted of trees.

(20) If X is constituted of finite trees and $Y \subseteq X$, then Y is constituted of finite trees.

(21) If X is constituted of decorated trees and $Y \subseteq X$, then Y is constituted of decorated trees.

One can verify that there exists a set which is finite, constituted of trees, constituted of finite trees, and non empty and there exists a set which is finite, constituted of decorated trees, and non empty.

Let us observe that every set which is constituted of finite trees is also constituted of trees.

Let X be a constituted of trees set. One can check that every subset of X is constituted of trees.

Let X be a constituted of finite trees set. Observe that every subset of X is constituted of finite trees.

Let X be a constituted of decorated trees set. One can check that every subset of X is constituted of decorated trees.

Let D be a constituted of trees non empty set. We see that the element of D is a tree.

Let D be a constituted of finite trees non empty set. We see that the element of D is a finite tree.

Let D be a constituted of decorated trees non empty set. We see that the element of D is a decorated tree.

Let us observe that Trees is constituted of trees.

One can check that there exists a subset of Trees which is constituted of finite trees and non empty.

Let us mention that FinTrees is constituted of finite trees.

Let D be a non empty set. A set is called a set of trees decorated with elements of D if:

(Def. 6) For every x such that $x \in$ it holds x is a tree decorated with elements of D.

Let D be a non empty set. Observe that every set of trees decorated with elements of D is constituted of decorated trees.

Let D be a non empty set. Note that there exists a set of trees decorated with elements of D which is finite and non empty.

Let D be a non empty set and let E be a non empty set of trees decorated with elements of D. We see that the element of E is a tree decorated with elements of D.

Let T be a tree and let D be a non empty set. Then D^T is a non empty set of trees decorated with elements of D. We see that the relation between T and D is a ParametrizedSubset of D.

Let T be a tree and let D be a non empty set. One can verify that every function from T into D is decorated tree-like.

Let D be a non empty set. The functor $\mathrm{Trees}(D)$ yields a set of trees decorated with elements of D and is defined as follows:

(Def. 7) For every tree T decorated with elements of D holds $T \in \mathrm{Trees}(D)$.

Let D be a non empty set. Observe that $\mathrm{Trees}(D)$ is non empty.

Let D be a non empty set. The functor $\mathrm{FinTrees}(D)$ yielding a set of trees decorated with elements of D is defined by:

(Def. 8) For every tree T decorated with elements of D holds $\mathrm{dom}\, T$ is finite iff $T \in \mathrm{FinTrees}(D)$.

Let D be a non empty set. One can verify that $\mathrm{FinTrees}(D)$ is non empty.

We now state the proposition

(22) For every non empty set D holds $\mathrm{FinTrees}(D) \subseteq \mathrm{Trees}(D)$.

3. Functions yielding trees

Let I_1 be a function. We say that I_1 is tree yielding if and only if:

(Def. 9) $\mathrm{rng}\, I_1$ is constituted of trees.

We say that I_1 is finite tree yielding if and only if:

(Def. 10) $\mathrm{rng}\, I_1$ is constituted of finite trees.

We say that I_1 is decorated tree yielding if and only if:

(Def. 11) $\mathrm{rng}\, I_1$ is constituted of decorated trees.

Next we state a number of propositions:

(23) $\emptyset$ is tree yielding, finite tree yielding, and decorated tree yielding.

(24) f is tree yielding iff for every x such that $x \in \mathrm{dom}\, f$ holds $f(x)$ is a tree.

(25) f is finite tree yielding iff for every x such that $x \in \mathrm{dom}\, f$ holds $f(x)$ is a finite tree.

(26) f is decorated tree yielding iff for every x such that $x \in \mathrm{dom}\, f$ holds $f(x)$ is a decorated tree.

(27) p is tree yielding and q is tree yielding iff $p \frown q$ is tree yielding.

(28) p is finite tree yielding and q is finite tree yielding iff $p \frown q$ is finite tree yielding.

(29) p is decorated tree yielding and q is decorated tree yielding iff $p \frown q$ is decorated tree yielding.

(30) $\langle x \rangle$ is tree yielding iff x is a tree.

(31) $\langle x \rangle$ is finite tree yielding iff x is a finite tree.

(32) $\langle x \rangle$ is decorated tree yielding iff x is a decorated tree.

(33) $\langle x, y \rangle$ is tree yielding iff x is a tree and y is a tree.

(34) $\langle x, y \rangle$ is finite tree yielding iff x is a finite tree and y is a finite tree.

(35) $\langle x, y \rangle$ is decorated tree yielding iff x is a decorated tree and y is a decorated tree.

(36) If $i \neq 0$, then $i \mapsto x$ is tree yielding iff x is a tree.

(37) If $i \neq 0$, then $i \mapsto x$ is finite tree yielding iff x is a finite tree.

(38) If $i \neq 0$, then $i \mapsto x$ is decorated tree yielding iff x is a decorated tree.

Let us observe that there exists a finite sequence which is tree yielding, finite tree yielding, and non empty and there exists a finite sequence which is decorated tree yielding and non empty.

One can verify that there exists a function which is tree yielding, finite tree yielding, and non empty and there exists a function which is decorated tree yielding and non empty.

One can check that every function which is finite tree yielding is also tree yielding.

Let D be a constituted of trees non empty set. One can verify that every finite sequence of elements of D is tree yielding.

Let p, q be tree yielding finite sequences. One can verify that $p \frown q$ is tree yielding.

Let D be a constituted of finite trees non empty set. One can verify that every finite sequence of elements of D is finite tree yielding.

Let p, q be finite tree yielding finite sequences. Note that $p \frown q$ is finite tree yielding.

Let D be a constituted of decorated trees non empty set. Observe that every finite sequence of elements of D is decorated tree yielding.

Let p, q be decorated tree yielding finite sequences. One can verify that $p \frown q$ is decorated tree yielding.

Let T be a tree. One can check that $\langle T \rangle$ is tree yielding and non empty. Let S be a tree. Observe that $\langle T, S \rangle$ is tree yielding and non empty.

Let n be a natural number and let T be a tree. Note that $n \mapsto T$ is tree yielding.

Let T be a finite tree. Observe that $\langle T \rangle$ is finite tree yielding. Let S be a finite tree. Observe that $\langle T, S \rangle$ is finite tree yielding.

Let n be a natural number and let T be a finite tree. Note that $n \mapsto T$ is finite tree yielding.

Let T be a decorated tree. Note that $\langle T \rangle$ is decorated tree yielding and non empty. Let S be a decorated tree. Note that $\langle T, S \rangle$ is decorated tree yielding and non empty.

Let n be a natural number and let T be a decorated tree. Note that $n \mapsto T$ is decorated tree yielding.

Next we state the proposition

(39) For every decorated tree yielding function f holds $\mathrm{dom}(\mathrm{dom}_\kappa f(\kappa)) = \mathrm{dom}\, f$ and $\mathrm{dom}_\kappa f(\kappa)$ is tree yielding.

Let p be a decorated tree yielding finite sequence. Note that $\mathrm{dom}_\kappa p(\kappa)$ is tree yielding and finite sequence-like.

One can prove the following proposition

(40) For every decorated tree yielding finite sequence p holds $\mathrm{len}(\mathrm{dom}_\kappa p(\kappa)) = \mathrm{len}\, p$.

4. Trees Decorated by Cartesian Product and Structure of Substitution

Let D, E be non empty sets. A tree decorated with elements of D and E is a tree decorated with elements of $[:D, E:]$. A set of trees decorated with elements of D and E is a set of trees decorated with elements of $[:D, E:]$.

Let T_1, T_2 be decorated trees. Observe that $\langle T_1, T_2 \rangle$ is decorated tree-like.

Let D_1, D_2 be non empty sets, let T_1 be a tree decorated with elements of D_1, and let T_2 be a tree decorated with elements of D_2. Then $\langle T_1, T_2 \rangle$ is a tree decorated with elements of D_1 and D_2.

Let D, E be non empty sets, let T be a tree decorated with elements of D, and let f be a function from D into E. Then $f \cdot T$ is a tree decorated with elements of E.

Let D_1, D_2 be non empty sets. Then $\pi_1(D_1 \times D_2)$ is a function from $[:D_1, D_2:]$ into D_1. Then $\pi_2(D_1 \times D_2)$ is a function from $[:D_1, D_2:]$ into D_2.

Let D_1, D_2 be non empty sets and let T be a tree decorated with elements of D_1 and D_2. The functor $T_{\mathbf{1}}$ yields a tree decorated with elements of D_1 and is defined as follows:

(Def. 12) $T_{\mathbf{1}} = \pi_1(D_1 \times D_2) \cdot T$.

The functor $T_{\mathbf{2}}$ yielding a tree decorated with elements of D_2 is defined as follows:

(Def. 13) $T_{\mathbf{2}} = \pi_2(D_1 \times D_2) \cdot T$.

Next we state two propositions:

(41) Let D_1, D_2 be non empty sets, T be a tree decorated with elements of D_1 and D_2, and t be an element of $\mathrm{dom}\, T$. Then $T(t)_{\mathbf{1}} = T_{\mathbf{1}}(t)$ and $T_{\mathbf{2}}(t) = T(t)_{\mathbf{2}}$.

(42) For all non empty sets D_1, D_2 and for every tree T decorated with elements of D_1 and D_2 holds $\langle T_{\mathbf{1}}, T_{\mathbf{2}} \rangle = T$.

Let T be a finite tree. Observe that $\mathrm{Leaves}(T)$ is finite and non empty.

Let T be a tree and let S be a non empty subset of T. We see that the element of S is an element of T.

Let T be a finite tree. We see that the leaf of T is an element of $\mathrm{Leaves}(T)$.

Let T be a finite tree. A tree is called a substitution of structure of T if:

(Def. 14) For every element t of it holds $t \in T$ or there exists a leaf l of T such that $l \prec t$.

Let T be a finite tree, let t be a leaf of T, and let S be a tree. Then T with-replacement(t, S) is a substitution of structure of T.

Let T be a finite tree. One can check that there exists a substitution of structure of T which is finite.

Let us consider n. A substitution of structure of n is a substitution of structure of the elementary tree of n.

The following propositions are true:

(43) Every tree is a substitution of structure of 0.

(44) For all trees T_1, T_2 such that T_1-level$(1) \subseteq T_2$-level(1) and for every n such that $\langle n \rangle \in T_1$ holds $T_1 \restriction \langle n \rangle = T_2 \restriction \langle n \rangle$ holds $T_1 \subseteq T_2$.

5. JOINING OF TREES

We now state four propositions:

(46)[1] For all trees T, T' and for every finite sequence p of elements of $\mathbb{N}$ such that $p \in \mathrm{Leaves}(T)$ holds $T \subseteq T$ with-replacement(p, T').

(47) For all decorated trees T, T' and for every element p of $\mathrm{dom}\, T$ holds $(T$ with-replacement$(p, T'))(p) = T'(\emptyset)$.

(48) For all decorated trees T, T' and for all elements p, q of $\mathrm{dom}\, T$ such that $p \not\preceq q$ holds $(T$ with-replacement$(p, T'))(q) = T(q)$.

(49) For all decorated trees T, T' and for every element p of $\mathrm{dom}\, T$ and for every element q of $\mathrm{dom}\, T'$ holds $(T$ with-replacement$(p, T'))(p \frown q) = T'(q)$.

Let T_1, T_2 be trees. Observe that $T_1 \cup T_2$ is non empty and tree-like.
One can prove the following proposition

(50) Let T_1, T_2 be trees and p be an element of $T_1 \cup T_2$. Then

(i) if $p \in T_1$ and $p \in T_2$, then $(T_1 \cup T_2) \restriction p = T_1 \restriction p \cup T_2 \restriction p$,

(ii) if $p \notin T_1$, then $(T_1 \cup T_2) \restriction p = T_2 \restriction p$, and

(iii) if $p \notin T_2$, then $(T_1 \cup T_2) \restriction p = T_1 \restriction p$.

Let us consider p. Let us assume that p is tree yielding. The functor $\overbrace{p}$ yielding a tree is defined as follows:

(Def. 15) $x \in \overbrace{p}$ iff $x = \emptyset$ or there exist n, q such that $n < \mathrm{len}\, p$ and $q \in p(n+1)$ and $x = \langle n \rangle \frown q$.

Let T be a tree. The functor $\overbrace{T}$ yields a tree and is defined as follows:

(Def. 16) $\overbrace{T} = \overbrace{\langle T \rangle}$.

Let T_1, T_2 be trees. The functor $\overbrace{T_1, T_2}$ yields a tree and is defined by:

(Def. 17) $\overbrace{T_1, T_2} = \overbrace{\langle T_1, T_2 \rangle}$.

We now state a number of propositions:

(51) If p is tree yielding, then $\langle n \rangle \frown q \in \overbrace{p}$ iff $n < \mathrm{len}\, p$ and $q \in p(n+1)$.

(52) If p is tree yielding, then $\overbrace{p}$-level$(1) = \{\langle n \rangle : n < \mathrm{len}\, p\}$ and for every n such that $n < \mathrm{len}\, p$ holds $\overbrace{p} \restriction \langle n \rangle = p(n+1)$.

(53) For all tree yielding finite sequences p, q such that $\overbrace{p} = \overbrace{q}$ holds $p = q$.

(54) For all tree yielding finite sequences p_1, p_2 and for every tree T holds $p \in T$ iff $\langle \mathrm{len}\, p_1 \rangle \frown p \in \overbrace{p_1 \frown \langle T \rangle \frown p_2}$.

(55) $\overbrace{\emptyset} =$ the elementary tree of 0.

(56) If p is tree yielding, then the elementary tree of $\mathrm{len}\, p \subseteq \overbrace{p}$.

(57) The elementary tree of $i = \overbrace{i \mapsto (\text{the elementary tree of } 0)}$.

[1] The proposition (45) has been removed.

(58) Let T be a tree and p be a tree yielding finite sequence. Then $\overbrace{p \frown \langle T \rangle} = (\overbrace{p} \cup$the elementary tree of $\mathrm{len}\, p+1)$ with-replacement$(\langle \mathrm{len}\, p \rangle, T)$.

(59) Let p be a tree yielding finite sequence. Then $\overbrace{p \frown \langle \text{the elementary tree of } 0 \rangle} = \overbrace{p} \cup$the elementary tree of $\mathrm{len}\, p+1$.

(60) For all tree yielding finite sequences p, q and for all trees T_1, T_2 holds $\overbrace{p \frown \langle T_1 \rangle \frown q} = \overbrace{p \frown \langle T_2 \rangle \frown q}$with-replacement$(\langle \mathrm{len}\, p \rangle, T_1)$.

(61) For every tree T holds $\overbrace{T} =$ (the elementary tree of 1) with-replacement$(\langle 0 \rangle, T)$.

(62) For all trees T_1, T_2 holds $\overbrace{T_1, T_2} =$ (the elementary tree of 2) with-replacement$(\langle 0 \rangle, T_1)$ with-replacement$(\langle 1 \rangle, T_2)$.

Let p be a finite tree yielding finite sequence. Observe that $\overbrace{p}$ is finite.
Let T be a finite tree. Observe that $\overbrace{T}$ is finite.
Let T_1, T_2 be finite trees. Observe that $\overbrace{T_1, T_2}$ is finite.
Next we state a number of propositions:

(63) For every tree T and for every set x holds $x \in \overbrace{T}$ iff $x = \emptyset$ or there exists p such that $p \in T$ and $x = \langle 0 \rangle \frown p$.

(64) For every tree T and for every finite sequence p holds $p \in T$ iff $\langle 0 \rangle \frown p \in \overbrace{T}$.

(65) For every tree T holds the elementary tree of $1 \subseteq \overbrace{T}$.

(66) For all trees T_1, T_2 such that $T_1 \subseteq T_2$ holds $\overbrace{T_1} \subseteq \overbrace{T_2}$.

(67) For all trees T_1, T_2 such that $\overbrace{T_1} = \overbrace{T_2}$ holds $T_1 = T_2$.

(68) For every tree T holds $\overbrace{T} \restriction \langle 0 \rangle = T$.

(69) For all trees T_1, T_2 holds $\overbrace{T_1}$ with-replacement$(\langle 0 \rangle, T_2) = \overbrace{T_2}$.

(70) $\overbrace{\text{the elementary tree of } 0} =$ the elementary tree of 1.

(71) Let T_1, T_2 be trees and x be a set. Then $x \in \overbrace{T_1, T_2}$ if and only if one of the following conditions is satisfied:

(i) $x = \emptyset$, or

(ii) there exists p such that $p \in T_1$ and $x = \langle 0 \rangle \frown p$ or $p \in T_2$ and $x = \langle 1 \rangle \frown p$.

(72) For all trees T_1, T_2 and for every finite sequence p holds $p \in T_1$ iff $\langle 0 \rangle \frown p \in \overbrace{T_1, T_2}$.

(73) For all trees T_1, T_2 and for every finite sequence p holds $p \in T_2$ iff $\langle 1 \rangle \frown p \in \overbrace{T_1, T_2}$.

(74) For all trees T_1, T_2 holds the elementary tree of $2 \subseteq \overbrace{T_1, T_2}$.

(75) For all trees T_1, T_2, W_1, W_2 such that $T_1 \subseteq W_1$ and $T_2 \subseteq W_2$ holds $\overbrace{T_1, T_2} \subseteq \overbrace{W_1, W_2}$.

(76) For all trees T_1, T_2, W_1, W_2 such that $\overbrace{T_1, T_2} = \overbrace{W_1, W_2}$ holds $T_1 = W_1$ and $T_2 = W_2$.

(77) For all trees T_1, T_2 holds $\overbrace{T_1, T_2} \restriction \langle 0 \rangle = T_1$ and $\overbrace{T_1, T_2} \restriction \langle 1 \rangle = T_2$.

(78) For all trees T, T_1, T_2 holds $\overbrace{T_1, T_2}$ with-replacement$(\langle 0 \rangle, T) = \overbrace{T, T_2}$ and $\overbrace{T_1, T_2}$ with-replacement$(\langle 1 \rangle, T) = \overbrace{T_1, T}$.

(79) $\overbrace{\text{the elementary tree of } 0, \text{the elementary tree of } 0} =$ the elementary tree of 2.

In the sequel w denotes a finite tree yielding finite sequence.

One can prove the following propositions:

(80) For every w such that for every finite tree t such that $t \in \operatorname{rng} w$ holds height $t \leq n$ holds height $\overbrace{w} \leq n+1$.

(81) For every finite tree t such that $t \in \operatorname{rng} w$ holds height $\overbrace{w} >$ height t.

(82) For every finite tree t such that $t \in \operatorname{rng} w$ and for every finite tree t' such that $t' \in \operatorname{rng} w$ holds height $t' \leq$ height t holds height $\overbrace{w} =$ height $t+1$.

(83) For every finite tree T holds height $\overbrace{T} =$ height $T+1$.

(84) For all finite trees T_1, T_2 holds height $\overbrace{T_1, T_2} = \max(\text{height } T_1, \text{height } T_2) + 1$.

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