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Transformations in Affine Spaces[1]

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Summary. Two classes of bijections of its point universe are correlated with every affine structure. The first class consists of the transformations, called formal isometries, which map every segment onto congruent segment, the second class consists of the automorphisms of such a structure. Each of these two classes of bijections forms a group for a given affine structure, if it satisfies a very weak axiom system (models of these axioms are called congruence spaces); formal isometries form a normal subgroup in the group of automorphism. In particular ordered affine spaces and affine spaces are congruence spaces; therefore formal isometries of these structures can be considered. They are called positive dilatations and dilatations, resp. For convenience the class of negative dilatations, transformations which map every "vector" onto parallel "vector", but with opposite sense, is singled out. The class of translations is distinguished as well. Basic facts concerning all these types of transformations are established, like rigidity, decomposition principle, introductory group-theoretical properties. At the end collineations of affine spaces and their properties are investigated; for affine planes it is proved that the class of collineations coincides with the class of bijections preserving lines.

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The articles [8], [4], [9], [11], [1], [10], [5], [6], [3], [2], and [7] provide the notation and terminology for this paper.

We adopt the following rules: A denotes a non empty set, a, b, x, y, z, t denote elements of A, and f, g, h denote permutations of A.

Let A be a set and let f, g be permutations of A. Then $g \cdot f$ is a permutation of A.

Next we state two propositions:

(2)[1] There exists x such that $f(x) = y$.

(4)[2] $f(x) = y$ iff $f^{-1}(y) = x$.

Let us consider A, f, g. The functor $f \setminus g$ yields a permutation of A and is defined as follows:

(Def. 1) $f \setminus g = g \cdot f \cdot g^{-1}$.

The scheme *EXPermutation* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

> There exists a permutation f of $\mathcal{A}$ such that for all elements x, y of $\mathcal{A}$ holds $f(x) = y$ iff $\mathcal{P}[x,y]$

[1]Supported by RPBP.III-24.C2.
[1] The proposition (1) has been removed.
[2] The proposition (3) has been removed.

provided the parameters satisfy the following conditions:

- For every element x of $\mathcal{A}$ there exists an element y of $\mathcal{A}$ such that $\mathcal{P}[x,y]$,
- For every element y of $\mathcal{A}$ there exists an element x of $\mathcal{A}$ such that $\mathcal{P}[x,y]$,
- For all elements x, y, x' of $\mathcal{A}$ such that $\mathcal{P}[x,y]$ and $\mathcal{P}[x',y]$ holds $x = x'$, and
- For all elements x, y, y' of $\mathcal{A}$ such that $\mathcal{P}[x,y]$ and $\mathcal{P}[x,y']$ holds $y = y'$.

One can prove the following propositions:

(9)[3] $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

(11)[4] $f \cdot \mathrm{id}_A = \mathrm{id}_A \cdot f$.

(13)[5] If $g \cdot f = h \cdot f$ or $f \cdot g = f \cdot h$, then $g = h$.

(16)[6] $f \cdot g \setminus h = (f \setminus h) \cdot (g \setminus h)$.

(17) $f^{-1} \setminus g = (f \setminus g)^{-1}$.

(18) $f \setminus g \cdot h = f \setminus h \setminus g$.

(19) $\mathrm{id}_A \setminus f = \mathrm{id}_A$.

(20) $f \setminus \mathrm{id}_A = f$.

(21) If $f(a) = a$, then $(f \setminus g)(g(a)) = g(a)$.

In the sequel R is a binary relation on $[:A, A:]$.

Let us consider A, f, R. We say that f is a formal isometry of R if and only if:

(Def. 2) For all x, y holds $\langle\langle x, y\rangle, \langle f(x), f(y)\rangle\rangle \in R$.

We now state four propositions:

(23)[7] If R is reflexive in $[:A, A:]$, then id_A is a formal isometry of R.

(24) If R is symmetric in $[:A, A:]$ and f is a formal isometry of R, then f^{-1} is a formal isometry of R.

(25) Suppose R is transitive in $[:A, A:]$ and f is a formal isometry of R and g is a formal isometry of R. Then $f \cdot g$ is a formal isometry of R.

(26) Suppose that

(i) for all a, b, x, y, z, t such that $\langle\langle x, y\rangle, \langle a, b\rangle\rangle \in R$ and $\langle\langle a, b\rangle, \langle z, t\rangle\rangle \in R$ and $a \neq b$ holds $\langle\langle x, y\rangle, \langle z, t\rangle\rangle \in R$,

(ii) for all x, y, z holds $\langle\langle x, x\rangle, \langle y, z\rangle\rangle \in R$,

(iii) f is a formal isometry of R, and

(iv) g is a formal isometry of R.

Then $f \cdot g$ is a formal isometry of R.

Let us consider A, let us consider f, and let us consider R. We say that f is an automorphism of R if and only if:

(Def. 3) For all x, y, z, t holds $\langle\langle x, y\rangle, \langle z, t\rangle\rangle \in R$ iff $\langle\langle f(x), f(y)\rangle, \langle f(z), f(t)\rangle\rangle \in R$.

The following propositions are true:

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[3] The propositions (5)–(8) have been removed.
[4] The proposition (10) has been removed.
[5] The proposition (12) has been removed.
[6] The propositions (14) and (15) have been removed.
[7] The proposition (22) has been removed.

(28)[8] id_A is an automorphism of R.

(29) If f is an automorphism of R, then f^{-1} is an automorphism of R.

(30) If f is an automorphism of R and g is an automorphism of R, then $g \cdot f$ is an automorphism of R.

(31) If R is symmetric in $[:A, A:]$ and transitive in $[:A, A:]$ and f is a formal isometry of R, then f is an automorphism of R.

(32) Suppose that

(i) for all a, b, x, y, z, t such that $\langle\langle x, y\rangle, \langle a, b\rangle\rangle \in R$ and $\langle\langle a, b\rangle, \langle z, t\rangle\rangle \in R$ and $a \neq b$ holds $\langle\langle x, y\rangle, \langle z, t\rangle\rangle \in R$,

(ii) for all x, y, z holds $\langle\langle x, x\rangle, \langle y, z\rangle\rangle \in R$,

(iii) R is symmetric in $[:A, A:]$, and

(iv) f is a formal isometry of R.

Then f is an automorphism of R.

(33) If f is a formal isometry of R and g is an automorphism of R, then $f \setminus g$ is a formal isometry of R.

In the sequel A_1 denotes a non empty affine structure.

Let us consider A_1 and let f be a permutation of the carrier of A_1. We say that f is a dilatation of A_1 if and only if:

(Def. 4) f is a formal isometry of the congruence of A_1.

In the sequel a, b denote elements of A_1.

One can prove the following proposition

(35)[9] Let f be a permutation of the carrier of A_1. Then f is a dilatation of A_1 if and only if for all a, b holds $a,b \upharpoonleft\upharpoonright f(a), f(b)$.

Let I_1 be a non empty affine structure. We say that I_1 is congruence space-like if and only if the conditions (Def. 5) are satisfied.

(Def. 5)(i) For all elements x, y, z, t, a, b of I_1 such that $x,y \upharpoonleft\upharpoonright a,b$ and $a,b \upharpoonleft\upharpoonright z,t$ and $a \neq b$ holds $x,y \upharpoonleft\upharpoonright z,t$,

(ii) for all elements x, y, z of I_1 holds $x,x \upharpoonleft\upharpoonright y,z$,

(iii) for all elements x, y, z, t of I_1 such that $x,y \upharpoonleft\upharpoonright z,t$ holds $z,t \upharpoonleft\upharpoonright x,y$, and

(iv) for all elements x, y of I_1 holds $x,y \upharpoonleft\upharpoonright x,y$.

One can verify that there exists a non empty affine structure which is strict and congruence space-like.

A congruence space is a congruence space-like non empty affine structure.

In the sequel C_1 denotes a congruence space.

The following three propositions are true:

(37)[10] $\mathrm{id}_{\text{the carrier of } C_1}$ is a dilatation of C_1.

(38) For every permutation f of the carrier of C_1 such that f is a dilatation of C_1 holds f^{-1} is a dilatation of C_1.

(39) Let f, g be permutations of the carrier of C_1. Suppose f is a dilatation of C_1 and g is a dilatation of C_1. Then $f \cdot g$ is a dilatation of C_1.

[8] The proposition (27) has been removed.
[9] The proposition (34) has been removed.
[10] The proposition (36) has been removed.

We follow the rules: O_1 denotes an ordered affine space and a, b, c, d, p, q, x, y, z denote elements of O_1.

We now state the proposition

(40) O_1 is congruence space-like.

In the sequel f, g are permutations of the carrier of O_1.

Let us consider O_1 and let f be a permutation of the carrier of O_1. We say that f is positive dilatation if and only if:

(Def. 6) f is a dilatation of O_1.

We introduce f is a positive dilatation as a synonym of f is positive dilatation.

One can prove the following proposition

(42)[11] Let f be a permutation of the carrier of O_1. Then f is a positive dilatation if and only if for all a, b holds $a, b \upuparrows f(a), f(b)$.

Let us consider O_1 and let f be a permutation of the carrier of O_1. We say that f is negative dilatation if and only if:

(Def. 7) For all a, b holds $a, b \upuparrows f(b), f(a)$.

We introduce f is a negative dilatation as a synonym of f is negative dilatation.

Next we state several propositions:

(44)[12] $\mathrm{id}_{\text{the carrier of } O_1}$ is a positive dilatation.

(45) For every permutation f of the carrier of O_1 such that f is a positive dilatation holds f^{-1} is a positive dilatation.

(46) Let f, g be permutations of the carrier of O_1. Suppose f is a positive dilatation and g is a positive dilatation. Then $f \cdot g$ is a positive dilatation.

(47) There exists no f which is a negative dilatation and a positive dilatation.

(48) If f is a negative dilatation, then f^{-1} is a negative dilatation.

(49) Suppose f is a positive dilatation and g is a negative dilatation. Then $f \cdot g$ is a negative dilatation and $g \cdot f$ is a negative dilatation.

Let us consider O_1 and let f be a permutation of the carrier of O_1. We say that f is dilatation if and only if:

(Def. 8) f is a formal isometry of λ(the congruence of O_1).

We introduce f is a dilatation as a synonym of f is dilatation.

We now state a number of propositions:

(51)[13] For every permutation f of the carrier of O_1 holds f is a dilatation iff for all a, b holds $a, b \parallel f(a), f(b)$.

(52) If f is a positive dilatation and a negative dilatation, then f is a dilatation.

(53) Let f be a permutation of the carrier of O_1. Suppose f is a dilatation. Then there exists a permutation f' of the carrier of $\Lambda(O_1)$ such that $f = f'$ and f' is a dilatation of $\Lambda(O_1)$.

(54) Let f be a permutation of the carrier of $\Lambda(O_1)$. Suppose f is a dilatation of $\Lambda(O_1)$. Then there exists a permutation f' of the carrier of O_1 such that $f = f'$ and f' is a dilatation.

[11] The proposition (41) has been removed.

[12] The proposition (43) has been removed.

[13] The proposition (50) has been removed.

(55) $\mathrm{id}_{\text{the carrier of } O_1}$ is a dilatation.

(56) If f is a dilatation, then f^{-1} is a dilatation.

(57) If f is a dilatation and g is a dilatation, then $f \cdot g$ is a dilatation.

(58) If f is a dilatation, then for all a, b, c, d holds $a,b \parallel c,d$ iff $f(a), f(b) \parallel f(c), f(d)$.

(59) If f is a dilatation, then for all a, b, c holds $\mathbf{L}(a,b,c)$ iff $\mathbf{L}(f(a), f(b), f(c))$.

(60) If f is a dilatation and $\mathbf{L}(x, f(x), y)$, then $\mathbf{L}(x, f(x), f(y))$.

(61) If $a,b \parallel c,d$, then $a,c \parallel b,d$ or there exists x such that $\mathbf{L}(a,c,x)$ and $\mathbf{L}(b,d,x)$.

(62) If f is a dilatation, then $f = \mathrm{id}_{\text{the carrier of } O_1}$ or for every x holds $f(x) \neq x$ iff for all x, y holds $x, f(x) \parallel y, f(y)$.

(63) If f is a dilatation and $f(a) = a$ and $f(b) = b$ and not $\mathbf{L}(a,b,x)$, then $f(x) = x$.

(64) If f is a dilatation and $f(a) = a$ and $f(b) = b$ and $a \neq b$, then $f = \mathrm{id}_{\text{the carrier of } O_1}$.

(65) If f is a dilatation and g is a dilatation and $f(a) = g(a)$ and $f(b) = g(b)$, then $a = b$ or $f = g$.

Let us consider O_1 and let f be a permutation of the carrier of O_1. We say that f is translation if and only if:

(Def. 9) f is a dilatation but $f = \mathrm{id}_{\text{the carrier of } O_1}$ or for every a holds $a \neq f(a)$.

We introduce f is a translation as a synonym of f is translation.

We now state a number of propositions:

(67)[14] If f is a dilatation, then f is a translation iff for all x, y holds $x, f(x) \parallel y, f(y)$.

(69)[15] If f is a translation and g is a translation and $f(a) = g(a)$ and not $\mathbf{L}(a, f(a), x)$, then $f(x) = g(x)$.

(70) If f is a translation and g is a translation and $f(a) = g(a)$, then $f = g$.

(71) If f is a translation, then f^{-1} is a translation.

(72) If f is a translation and g is a translation, then $f \cdot g$ is a translation.

(73) If f is a translation, then f is a positive dilatation.

(74) If f is a dilatation and $f(p) = p$ and p is a midpoint of q, $f(q)$ and not $\mathbf{L}(p,q,x)$, then p is a midpoint of x, $f(x)$.

(75) If f is a dilatation and $f(p) = p$ and p is a midpoint of q, $f(q)$ and $q \neq p$, then p is a midpoint of x, $f(x)$.

(76) If f is a dilatation and $f(p) = p$ and $q \neq p$ and p is a midpoint of q, $f(q)$ and not $\mathbf{L}(p,x,y)$, then $x,y \upharpoonleft\!\!\downharpoonright f(y), f(x)$.

(77) If f is a dilatation and $f(p) = p$ and $q \neq p$ and p is a midpoint of q, $f(q)$ and $\mathbf{L}(p,x,y)$, then $x,y \upharpoonleft\!\!\downharpoonright f(y), f(x)$.

(78) If f is a dilatation and $f(p) = p$ and $q \neq p$ and p is a midpoint of q, $f(q)$, then f is a negative dilatation.

(79) If f is a dilatation and $f(p) = p$ and for every x holds $p,x \upharpoonleft\!\!\downharpoonright p, f(x)$, then for all y, z holds $y,z \upharpoonleft\!\!\downharpoonright f(y), f(z)$.

[14] The proposition (66) has been removed.
[15] The proposition (68) has been removed.

(80) If f is a dilatation, then f is a positive dilatation and a negative dilatation.

We use the following convention: A_2 denotes an affine space and a, b, c, d, d_1, d_2, x, y, z, t denote elements of A_2.

Next we state two propositions:

(82)[16] A_2 is congruence space-like.

(83) $\Lambda(O_1)$ is a congruence space.

In the sequel f, g are permutations of the carrier of A_2.

Let us consider A_2 and let us consider f. We say that f is dilatation if and only if:

(Def. 10) f is a dilatation of A_2.

We introduce f is a dilatation as a synonym of f is dilatation.

We now state a number of propositions:

(85)[17] f is a dilatation iff for all a, b holds $a,b \upuparrows f(a), f(b)$.

(86) $\mathrm{id}_{\text{the carrier of } A_2}$ is a dilatation.

(87) If f is a dilatation, then f^{-1} is a dilatation.

(88) If f is a dilatation and g is a dilatation, then $f \cdot g$ is a dilatation.

(89) If f is a dilatation, then for all a, b, c, d holds $a,b \upuparrows c,d$ iff $f(a), f(b) \upuparrows f(c), f(d)$.

(90) If f is a dilatation, then for all a, b, c holds $\mathbf{L}(a,b,c)$ iff $\mathbf{L}(f(a), f(b), f(c))$.

(91) If f is a dilatation and $\mathbf{L}(x, f(x), y)$, then $\mathbf{L}(x, f(x), f(y))$.

(92) If $a,b \upuparrows c,d$, then $a,c \upuparrows b,d$ or there exists x such that $\mathbf{L}(a,c,x)$ and $\mathbf{L}(b,d,x)$.

(93) If f is a dilatation, then $f = \mathrm{id}_{\text{the carrier of } A_2}$ or for every x holds $f(x) \neq x$ iff for all x, y holds $x, f(x) \upuparrows y, f(y)$.

(94) If f is a dilatation and $f(a) = a$ and $f(b) = b$ and not $\mathbf{L}(a,b,x)$, then $f(x) = x$.

(95) If f is a dilatation and $f(a) = a$ and $f(b) = b$ and $a \neq b$, then $f = \mathrm{id}_{\text{the carrier of } A_2}$.

(96) If f is a dilatation and g is a dilatation and $f(a) = g(a)$ and $f(b) = g(b)$, then $a = b$ or $f = g$.

(97) If not $\mathbf{L}(a,b,c)$ and $a,b \upuparrows c,d_1$ and $a,b \upuparrows c,d_2$ and $a,c \upuparrows b,d_1$ and $a,c \upuparrows b,d_2$, then $d_1 = d_2$.

Let us consider A_2 and let us consider f. We say that f is translation if and only if:

(Def. 11) f is a dilatation but $f = \mathrm{id}_{\text{the carrier of } A_2}$ or for every a holds $a \neq f(a)$.

We introduce f is a translation as a synonym of f is translation.

Next we state several propositions:

(99)[18] $\mathrm{id}_{\text{the carrier of } A_2}$ is a translation.

(100) If f is a dilatation, then f is a translation iff for all x, y holds $x, f(x) \upuparrows y, f(y)$.

(102)[19] If f is a translation and g is a translation and $f(a) = g(a)$ and not $\mathbf{L}(a, f(a), x)$, then $f(x) = g(x)$.

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[16] The proposition (81) has been removed.
[17] The proposition (84) has been removed.
[18] The proposition (98) has been removed.
[19] The proposition (101) has been removed.

(103) If f is a translation and g is a translation and $f(a) = g(a)$, then $f = g$.

(104) If f is a translation, then f^{-1} is a translation.

(105) If f is a translation and g is a translation, then $f \cdot g$ is a translation.

Let us consider A_2 and let us consider f. We say that f is collineation if and only if:

(Def. 12) f is an automorphism of the congruence of A_2.

We introduce f is a collineation as a synonym of f is collineation.
Next we state three propositions:

(107)[20] f is a collineation iff for all x, y, z, t holds $x,y \upuparrows z,t$ iff $f(x), f(y) \upuparrows f(z), f(t)$.

(108) If f is a collineation, then $\mathbf{L}(x,y,z)$ iff $\mathbf{L}(f(x), f(y), f(z))$.

(109) Suppose f is a collineation and g is a collineation. Then f^{-1} is a collineation and $f \cdot g$ is a collineation and $\mathrm{id}_{\text{the carrier of } A_2}$ is a collineation.

In the sequel A, C, K denote subsets of A_2.
Next we state several propositions:

(110) If $a \in A$, then $f(a) \in f^\circ A$.

(111) $x \in f^\circ A$ iff there exists y such that $y \in A$ and $f(y) = x$.

(112) If $f^\circ A = f^\circ C$, then $A = C$.

(113) If f is a collineation, then $f^\circ \mathrm{Line}(a,b) = \mathrm{Line}(f(a), f(b))$.

(114) If f is a collineation and K is a line, then $f^\circ K$ is a line.

(115) If f is a collineation and A // C, then $f^\circ A$ // $f^\circ C$.

For simplicity, we adopt the following rules: A_3 is an affine plane, A, K are subsets of A_3, p, x are elements of A_3, and f is a permutation of the carrier of A_3.
The following propositions are true:

(116) If for every A such that A is a line holds $f^\circ A$ is a line, then f is a collineation.

(117) Suppose f is a collineation and K is a line and for every x such that $x \in K$ holds $f(x) = x$ and $p \notin K$ and $f(p) = p$. Then $f = \mathrm{id}_{\text{the carrier of } A_3}$.

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