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Some Properties of Real Numbers Operations: min, max, square, and square root[1]

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Summary. We define the following operations on real numbers: $max(x,y)$, $min(x,y)$, x^2, $\sqrt{x}$. We prove basic properties of introduced operations. A number of auxiliary theorems absent in [2] and [3] is proved.

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The articles [4], [6], [1], [5], and [3] provide the notation and terminology for this paper.

In this paper a, b, x, y, z denote real numbers.

The following propositions are true:

(2)[1] If $1 < x$, then $\frac{1}{x} < 1$.

(11)[2] If $x < y$, then $0 < y - x$.

(12) If $x \leq y$, then $0 \leq y - x$.

(19)[3] If $0 \leq x$ and $0 \leq y$, then $0 \leq x \cdot y$.

(20) If $x \leq 0$ and $y \leq 0$, then $0 \leq x \cdot y$.

(21) If $0 < x$ and $0 < y$, then $0 < x \cdot y$.

(22) If $x < 0$ and $y < 0$, then $0 < x \cdot y$.

(23) If $0 \leq x$ and $y \leq 0$, then $x \cdot y \leq 0$.

(24) If $0 < x$ and $y < 0$, then $x \cdot y < 0$.

(25) If $0 \leq x \cdot y$, then $0 \leq x$ and $0 \leq y$ or $x \leq 0$ and $y \leq 0$.

(26) If $0 < x \cdot y$, then $0 < x$ and $0 < y$ or $x < 0$ and $y < 0$.

(27) If $0 \leq a$ and $0 \leq b$, then $0 \leq \frac{a}{b}$.

(29)[4] If $0 < x$, then $y - x < y$.

[1]Supported by RPBP.III-24.C1.

[1] The proposition (1) has been removed.

[2] The propositions (3)–(10) have been removed.

[3] The propositions (13)–(18) have been removed.

[4] The proposition (28) has been removed.

The scheme *RealContinuity* concerns two unary predicates $\mathcal{P}$, $\mathcal{Q}$, and states that:

There exists z such that for all x, y such that $\mathcal{P}[x]$ and $\mathcal{Q}[y]$ holds $x \leq z$ and $z \leq y$

provided the following condition is met:

- For all x, y such that $\mathcal{P}[x]$ and $\mathcal{Q}[y]$ holds $x \leq y$.

Let us consider x, y. The functor $\min(x,y)$ yields a real number and is defined by:

(Def. 1) $\min(x,y) = \begin{cases} x, & \text{if } x \leq y, \\ y, & \text{otherwise.} \end{cases}$

Let us observe that the functor $\min(x,y)$ is commutative and idempotent. The functor $\max(x,y)$ yielding a real number is defined as follows:

(Def. 2) $\max(x,y) = \begin{cases} x, & \text{if } y \leq x, \\ y, & \text{otherwise.} \end{cases}$

Let us notice that the functor $\max(x,y)$ is commutative and idempotent.

Let x, y be elements of $\mathbb{R}$. Then $\min(x,y)$ is an element of $\mathbb{R}$. Then $\max(x,y)$ is an element of $\mathbb{R}$.

The following propositions are true:

(34)[5] $\min(x,y) = \frac{(x+y)-|x-y|}{2}$.

(35) $\min(x,y) \leq x$.

(38)[6] $\min(x,y) = x$ or $\min(x,y) = y$.

(39) $x \leq y$ and $x \leq z$ iff $x \leq \min(y,z)$.

(40) $\min(x,\min(y,z)) = \min(\min(x,y),z)$.

(45)[7] $\max(x,y) = \frac{x+y+|x-y|}{2}$.

(46) $x \leq \max(x,y)$.

(49)[8] $\max(x,y) = x$ or $\max(x,y) = y$.

(50) $y \leq x$ and $z \leq x$ iff $\max(y,z) \leq x$.

(51) $\max(x,\max(y,z)) = \max(\max(x,y),z)$.

(53)[9] $\min(x,y) + \max(x,y) = x + y$.

(54) $\max(x,\min(x,y)) = x$.

(55) $\min(x,\max(x,y)) = x$.

(56) $\min(x,\max(y,z)) = \max(\min(x,y),\min(x,z))$.

(57) $\max(x,\min(y,z)) = \min(\max(x,y),\max(x,z))$.

Let us consider x. The functor $x^{\mathbf{2}}$ is defined as follows:

(Def. 3) $x^{\mathbf{2}} = x \cdot x$.

Let us consider x. Note that $x^{\mathbf{2}}$ is real.

Let x be an element of $\mathbb{R}$. Then $x^{\mathbf{2}}$ is an element of $\mathbb{R}$.

We now state a number of propositions:

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[5] The propositions (30)–(33) have been removed.
[6] The propositions (36) and (37) have been removed.
[7] The propositions (41)–(44) have been removed.
[8] The propositions (47) and (48) have been removed.
[9] The proposition (52) has been removed.

(59)[10] $1^2 = 1$.

(60) $0^2 = 0$.

(61) $a^2 = (-a)^2$.

(62) $|a|^2 = a^2$.

(63) $(a+b)^2 = a^2 + 2 \cdot a \cdot b + b^2$.

(64) $(a-b)^2 = (a^2 - 2 \cdot a \cdot b) + b^2$.

(65) $(a+1)^2 = a^2 + 2 \cdot a + 1$.

(66) $(a-1)^2 = (a^2 - 2 \cdot a) + 1$.

(67) $(a-b) \cdot (a+b) = a^2 - b^2$ and $(a+b) \cdot (a-b) = a^2 - b^2$.

(68) $(a \cdot b)^2 = a^2 \cdot b^2$.

(69) $(\frac{a}{b})^2 = \frac{a^2}{b^2}$.

(70) If $a^2 - b^2 \neq 0$, then $\frac{1}{a+b} = \frac{a-b}{a^2-b^2}$.

(71) If $a^2 - b^2 \neq 0$, then $\frac{1}{a-b} = \frac{a+b}{a^2-b^2}$.

(72) $0 \leq a^2$.

(73) If $a^2 = 0$, then $a = 0$.

(74) If $0 \neq a$, then $0 < a^2$.

(75) If $0 < a$ and $a < 1$, then $a^2 < a$.

(76) If $1 < a$, then $a < a^2$.

(77) If $0 \leq x$ and $x \leq y$, then $x^2 \leq y^2$.

(78) If $0 \leq x$ and $x < y$, then $x^2 < y^2$.

Let us consider a. Let us assume that $0 \leq a$. The functor $\sqrt{a}$ yielding a real number is defined as follows:

(Def. 4) $0 \leq \sqrt{a}$ and $\sqrt{a}^2 = a$.

Let a be an element of $\mathbb{R}$. Then $\sqrt{a}$ is an element of $\mathbb{R}$.

Next we state a number of propositions:

(82)[11] $\sqrt{0} = 0$.

(83) $\sqrt{1} = 1$.

(84) $1 < \sqrt{2}$.

(85) $\sqrt{4} = 2$.

(86) $\sqrt{2} < 2$.

(89)[12] If $0 \leq a$, then $\sqrt{a^2} = a$.

(90) If $a \leq 0$, then $\sqrt{a^2} = -a$.

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[10] The proposition (58) has been removed.
[11] The propositions (79)–(81) have been removed.
[12] The propositions (87) and (88) have been removed.

(91) $\sqrt{a^2} = |a|$.

(92) If $0 \leq a$ and $\sqrt{a} = 0$, then $a = 0$.

(93) If $0 < a$, then $0 < \sqrt{a}$.

(94) If $0 \leq x$ and $x \leq y$, then $\sqrt{x} \leq \sqrt{y}$.

(95) If $0 \leq x$ and $x < y$, then $\sqrt{x} < \sqrt{y}$.

(96) If $0 \leq x$ and $0 \leq y$ and $\sqrt{x} = \sqrt{y}$, then $x = y$.

(97) If $0 \leq a$ and $0 \leq b$, then $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$.

(98) If $0 \leq a \cdot b$, then $\sqrt{a \cdot b} = \sqrt{|a|} \cdot \sqrt{|b|}$.

(99) If $0 \leq a$ and $0 \leq b$, then $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

(100) If $0 < \frac{a}{b}$, then $\sqrt{\frac{a}{b}} = \frac{\sqrt{|a|}}{\sqrt{|b|}}$.

(101) If $0 < a$, then $\sqrt{\frac{1}{a}} = \frac{1}{\sqrt{a}}$.

(102) If $0 < a$, then $\frac{\sqrt{a}}{a} = \frac{1}{\sqrt{a}}$.

(103) If $0 < a$, then $\frac{a}{\sqrt{a}} = \sqrt{a}$.

(104) If $0 \leq a$ and $0 \leq b$, then $(\sqrt{a} - \sqrt{b}) \cdot (\sqrt{a} + \sqrt{b}) = a - b$.

(105) If $0 \leq a$ and $0 \leq b$ and $a \neq b$, then $\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$.

(106) If $0 \leq b$ and $b < a$, then $\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$.

(107) If $0 \leq a$ and $0 \leq b$ and $a \neq b$, then $\frac{1}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{a-b}$.

(108) If $0 \leq b$ and $b < a$, then $\frac{1}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{a-b}$.

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